Wigner’s little group and decomposition of Lorentz transformations

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It is shown how an arbitrary Lorentz transformation can be expressed in terms of elements of Wigner’s little group and its cosets. This yields a natural parametrization for the little group, while its coset members turn out to be helicity-preserving transformations. The associated Wigner angle and its relation to the actual change in helicity are discussed. Finally, the extension to zero-mass particles shows how the little group becomes a gauge transformation in that limit.

I. INTRODUCTION

In a 1957 paper,¹ Wigner discusses the role of the Lorentz transformations in understanding the internal symmetries of space-time, as they refer to a particle’s four-momentum and spin state.

In particular, there are certain transformations that leave such quantities invariant after their application. By definition, those that leave a four-momentum invariant form the so-called Wigner’s little group for that momentum. Other transformations change the momentum but leave the helicity invariant, as, for instance, any rotation would do. A generic transformation will then affect both the four-momentum and the helicity. In fact, the resulting state of these two quantities will depend on the particle’s mass, as well as on the “path” it follows, as it is boosted and rotated in the momentum space.

The theorem to be shown in Sec. II states that an arbitrary Lorentz transformation may be written as the product of a member of the little group times a member of its (left or right) coset. That coset member is a transformation that leaves the helicity invariant. Expressions for the parameters of the little group and the coset members can then be calculated. Working in the O(3) formalism (Sec. III) we find an expression for the Wigner angle²–⁴ related to the little-group member; subsequently we calculate the actual amount of spin rotation relative to the momentum direction now using the spinor representation. It is interesting to see how the little-group transformation changes form and meaning when it is applied to massless particles.⁵ In Sec. IV we find that it becomes a gauge transformation matrix; its different contents in the two representations are then discussed.

The above theorem has been discussed previously in the literature,²–⁴ but this was only done for special cases of the parameters involved. This paper aims to generalize those results in an attempt to unify them.

II. COSET DECOMPOSITION OF THE LORENTZ GROUP

Lorentz transformations can be visualized in the environment of the three-dimensional momentum space. A momentum state is specified by the triad (pₓ, pᵧ, p₂) in that space. For the sake of simplicity the spatial directions will be referred to as x, y, z instead of pₓ, pᵧ, p₂, respectively. Boosts are performed by vector additions and rotations refer to the origin. Spin and its rotations, helicity, and other features related to spin are not shown in such a picture.

Considering an arbitrary boost Bₘ(β), of boost parameter β, it is interesting to examine if it can be decomposed in parts, each of which keeps either the helicity or the four-momentum constant. The boost changes the momentum state of a particle in a well-defined way, bringing it from p to p' (Fig. 1). It is not obvious, though, how the helicity changes in the process. For simplicity we choose the initial state (p) to lie along the z axis, and then that axis together with Bₘ define the x-y plane (no loss of generality). The angle φ can have any value between 0° and 180°, measured from the z axis. Using the four-vector representation, we first construct the state p by applying a boost Aₘ(α) on the unit-mass particle, initially defined to be at rest and in the positive-helicity state:

\[ Aₘ(α) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & α/α & 0 \\ 1 & 0 & α/α & 1 \end{bmatrix} \equiv p \]  \hspace{1cm} (1)

[where α is the boost velocity of Aₘ, and α = (1 - α²)⁻¹/₂; no change of helicity has occurred.]

After that, we can apply Bₘ(β) on state p. Here

\[ Bₘ(β) = \begin{bmatrix} 1 + (γ - 1)β² / β₂ & 0 & (γ - 1)β₁β₂ / β₂ \\ 0 & 1 & 0 \\ (γ - 1)β₂β₁ / β₂ & 0 & 1 + (γ - 1)β₂β₁ / β₂ \\ β₁γ & 0 & β₂γ & γ \end{bmatrix} \]  \hspace{1cm} (2)

where β₁ = β sin(φ), β₂ = β cos(φ), and γ = (1 - β²)⁻¹/₂.

The resulting momentum state (point p') can be reached by other paths on the x-y plane. All of them will have the same effect on four-momentum, but they will affect helicity differently. A transformation that preserves helicity is, for example, a rotation R(θ) applied in succession to a boost in the z direction Bₘ(ε), in such a way that we again reach p'.
FIG. 1. The two-dimensional momentum space for Sec. II. A state \( p \) is first created by boost \( A_\alpha \), and then boosted by \( B_{\phi} \) to \( p' \). Here \( B_{\phi} \) can be decomposed in a helicity-preserving part \( R(\phi)B_{\phi} \) and a little-group member. The latter is momentum preserving by definition (therefore absent from momentum space), but gives an additional rotation to the particle spin.

Eventually,

\[
\begin{align*}
B_{\phi}(\epsilon) &= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1/e & e/\epsilon \\
0 & 0 & \epsilon/e & 1/\epsilon \\
\cos(\phi) & 0 & \sin(\phi) & 0 \\
0 & 1 & 0 & 0 \\
-\sin(\phi) & 0 & \cos(\phi) & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\end{align*}
\]

with \( \epsilon = (\text{boost velocity of } B_{\phi}) \) and \( \epsilon = (1 - \epsilon^2)^{1/2} \).

Both transformations \( R(\phi) \) and \( B_{\phi}(\epsilon) \) are helicity preserving, so if the particle follows the second path, its spin will not change its orientation relative to the momentum. A direct way to show that the two paths \( B_{\phi} \) and \( RB_{\phi} \) are not totally equivalent, is to compute the “closed-loop” matrix product \( B_{\phi}^{-1}RB_{\phi} \). Thus

\[
D = B_{\phi}^{-1}R(\phi)B_{\phi}
\]

\[
\begin{bmatrix}
T - 1 & -u & au \\
u & 1 & 0 \\
u & 0 & 1 - u^2/T
\end{bmatrix}
\]

where

\[
F = \sqrt{(\alpha \beta + 1)^2 - \alpha^2 \beta^2}, \quad u = -\beta_x/F,
\]
\[
T = 1 + (\alpha + \beta_x)/F.
\]

In general, matrix (4) is not the identity matrix. We can say that \( D \) is “equivalent to \( B_{\phi} \) with respect to helicity” (both result in the same change of helicity when applied to \( p \)). Since \( D \) maps a point of the momentum space in itself, or keeps its four-momentum invariant, it is a member of the little group for \( p \), by definition.

We see that Wigner’s little group can be parametrized using \( \alpha \) (boost parameter of \( p \)), \( \beta, \phi \) (parameters of the helicity-equivalent boost \( B_{\phi} \)) as the relevant quantities \( \epsilon, \phi \) are functions of those three variables as follows

\[
\begin{align*}
\epsilon &= -\alpha + \gamma \xi (\alpha \beta + 1) \\
\sin(\phi) &= \beta_x \frac{\alpha \beta + 1 + 1/\gamma}{(\alpha \beta + 1)(\gamma + 1)};
\end{align*}
\]

also see Fig. 2 for the variation of \( \epsilon \) with \( \alpha, \phi \).

The above choice of parameters is justified if we consider the case where \( \epsilon = 0 \) (no difference between initial and final boost parameters). In this case the parametrization of \( D \) becomes the so-called Eulerian parametrization of Wigner’s little group, which is now locally isomorphic to \( O(3) \). The \( D \) is expressed in terms of the \( z \) boost it acts on, and an angle around the \( y \) direction.

Now we can easily prove the theorem. We solve the above matrix equation for \( B_{\phi} \):

\[
B_{\phi}(\beta) = R(\phi)B_{\phi}(\epsilon)D^{-1},
\]

from this we see that arbitrary \( B_{\phi} \) has been broken down in two pieces: \( RB_{\phi} \) is helicity invariant and only changes the four-momentum, while \( D^{-1} \) does not affect the momentum state and only transforms the helicity. In the above equation the helicity-invariant part is a member of the left coset of the little group, but the expression can be rewritten in terms of a right coset member \( [B_{\phi} = (RB_{\phi}D^{-1}R^{-1}B_{\phi}^{-1})(B_{\phi}R)] \).

III. WIGNER ANGLE AND CHANGE OF HELICITY

In this section we go on deriving some results using the formalism of Sec. II. Then we see how this approach fits to the spinor representation.

A. O(3) formalism

In Eq. (6) the only terms that bring about a change in the momentum \( p \) are the helicity-preserving \( RB_{\phi} \); the action of \( D^{-1} \) does not apply on the kinematics of the particle. Therefore, in order to change helicity, \( D^{-1} \) can only rotate the spin. A measure of this rotation can be obtained if we consider an alternative momentum-preserving transformation: \( A_zR_wA_z^{-1} \) and set it equal to \( D \). Essentially we move the particle to the rest frame, rotate its spin there, and then restitute its momentum. Then

\[
R_w(\Phi_w) = A_z^{-1}DA_z.
\]

The angle \( \Phi_w \) is the so-called Wigner angle corresponding to the little-group element \( D \). However, \( R_w(\Phi_w) \) is not the actual rotation of the spin as the particle is boosted from
The matrices of Sec. II will take on the form

\[ B_\phi(\beta) = \begin{pmatrix} C + S_n & S_n_x \\ S_n_x & C - S_n \end{pmatrix}, \quad R(\vartheta) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \]

\[ A_z = \begin{pmatrix} N & 0 \\ 0 & 1/N \end{pmatrix}, \]

with \( C = (\gamma - 1)^{1/2}, \quad S = (\gamma + 1)^{1/2}, \quad n_x = \sin(\phi), \quad n_y = \cos(\phi), \quad c = \cos(\vartheta/2), \quad s = \sin(\vartheta/2), \) and \( N = [(1 + \alpha)/(1 - \alpha)]^{1/4}. \) Then the transformation \( D \) equals

\[ D^{(+)} = \begin{pmatrix} \sqrt{T/2} & auN^{2} \\ -auN^{2} & \sqrt{T/2} \end{pmatrix} \]

(9)

where \( u,T \) are the expressions defined in Sec. II.

We could now go on and repeat the calculations of Sec. III A to compute the Wigner angle. Instead, though, we can easily find the angle formed by the spin and momentum directions. When the particle is at the origin, it can be represented by a spinor of the form \( \chi' \pm \) (assume a polarization along the \( z \) axis). After \( A_z(\alpha) \) is applied, the resulting matrix, \( \chi' \pm (p) = A_z(\alpha)\chi' \pm = N^{-1}\chi' \pm, \) is boosted by \( B_\phi \) to become

\[ \chi' \pm (p') = B_\phi(\beta)\chi' \pm (p) = N^{-1}\chi' \pm, \]

where \( \chi' \pm \) is

\[ \chi' = \begin{pmatrix} (1 + b)\sqrt{2} & -\cos(\phi)(1 + b)^{1/2} \\ 2b & -\sin(\phi)(1 - b)^{1/2} \end{pmatrix}, \]

\[ \chi' = \begin{pmatrix} -\sin(\phi) & \cos(\phi)(1 - b)^{1/2} \\ -\sin(\phi) & -\cos(\phi)(1 - b)^{1/2} \end{pmatrix}, \]

with \( b = (1 - \beta^2)^{1/2} = 1/\gamma. \)

The above spinors can also be obtained (in normalized form) using a pure rotation by an angle \( \omega \pm \) around the \( y \) axis. The angles of rotation for dotted and undotted spinors will equal

\[ \tan\left(\frac{\omega_\pm}{2}\right) = \frac{1 + b \pm \beta \cos(\phi)}{\pm \beta \sin(\phi)}, \]

(10)

\[ \tan\left(\frac{\omega_\pm}{2}\right) = \frac{\pm \beta \sin(\phi)}{1 + b \mp \beta \cos(\phi)}, \]

where the upper sign refers to undotted spinors and the lower to dotted ones.

So the angle \( \delta \) between the spin and momentum at the point \( p' \) (in other words, the change in helicity resulting from \( B_\phi \)) will be

\[ \delta = \vartheta - \omega \pm, \]

(11)

\( \vartheta \) still having the same value as the one given by (5).

These angles are plotted versus \( \alpha \) and \( \phi \) in Fig. 4 for the case of undotted spinors. In Fig. 5 their relative magnitudes are shown in comparison to \( \Phi_w \).

IV. THE MASSLESS-PARTICLE LIMIT

So far our approach has been mass independent. It is interesting to check what happens if, keeping the moment-
two others mixes with the opposite-polarization spinor:

\[
\begin{align*}
D^{(+)} \chi_+ &= \chi_+ + u \chi_-, \\
D^{(-)} \chi_- &= \chi_- - u \chi_+,
\end{align*}
\]

(12)

Physically the invariant set can be polarized neutrinos. For the two others, helicity is not preserved under the action of the little group as we go to the zero-mass limit; they are not forced to align. These two extra degrees of freedom correspond to the gauge degrees of freedom in the case of photons.

V. CONCLUDING REMARKS

In this paper, the general form of a decomposition theorem for Lorentz transformations was shown. It turns out that a boost can be decomposed in two factors: one keeps the four-momentum invariant (is a member of the little group for that momentum), the other one preserves helicity. In this way we are led to a natural parametrization of Wigner’s little group. A member of that group, \( D \), is expressed in terms of the boost it acts on and the parameters (boost \( \beta \), relative angle \( \phi \)) of a boost that is equivalent to \( D \) with respect to helicity. Little-group members also have a Wigner’s angle associated with them. This gives a measure of the amount of spin rotation when \( D \) acts on the particle (momentum and spin) state. Here we calculated this angle as well as the actual change in helicity (using the spinor representation). Finally, it was shown that, at the zero-mass limit (case of photons or neutrinos), the little group turns into a gauge transformation matrix. Only one out of two spinors in each pair aligned with the momentum in this limit. The non-alignment of the remaining spinors gives rise to the particle’s gauge degrees of freedom.

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