Phys 783 - Plasma Kinetic Theory
Assignment # 4
Due Monday, February 23, 2009

Reminders: Show your work! Include references on your submitted version. Write legibly!

1. **Deriving the Bulk Energy Equation from the Vlasov Equation**

   (a) Derive the bulk energy equation from the Vlasov equation by taking the second moment, *i.e.,* multiply by $mv^2/2$ and integrate over velocity space. The result is

   $$\frac{\partial}{\partial t}\left(\frac{1}{2}mnu^2 + \frac{3}{2}P\right) + \nabla\cdot\left(\frac{1}{2}mnu^2\mathbf{u} + \frac{3}{2}P\mathbf{u} + \mathbf{u}\cdot\mathbf{P} + \mathbf{q}\right) = qn\mathbf{E}\cdot\mathbf{u}.$$

   where $P_{ij} = m\int d^3v' v'_i v'_j f$ are the pressure tensor elements, $P = P_{ii}/3$ is the scalar pressure, and $\mathbf{q} = \int d^3v' \frac{1}{2}mv'^{\,2}\mathbf{v}' f$ is the heat flux.

   (b) We argued in class that the above expression can be written as

   $$\frac{\partial}{\partial t}\left(\frac{3}{2}P\right) + \nabla\cdot\left(\frac{3}{2}P\mathbf{u}\right) + \nabla\cdot(\mathbf{u}\cdot\mathbf{P} + \mathbf{q}) - (\nabla\cdot\mathbf{P})\cdot\mathbf{u} = 0$$

   by using the continuity and momentum equations to isolate the thermal energy. (You are **not** being asked to show this!) Compare the importance of the heat flux term to the pressure term using a scaling analysis to see when the two terms dominate. In other words, let u be a typical velocity of the motion and v_{th} be a typical velocity perturbation, and estimate

   $$\frac{|\mathbf{u}\cdot\mathbf{P}|}{|\mathbf{q}|}.$$

   Show that the heat flux is negligible if $u \gg v_{th}$ (the system is adiabatic), and the heat flux dominates if $u \ll v_{th}$ (the system is isothermal). See pg. 46 of Bellan for some help. Note, this means that things get difficult if $u \sim v_{th}$!

2. **Electron Plasma Waves and Ion Acoustic Waves - Fluid Theory**

   In class, we used fluid theory to derive the dispersion relation for electron plasma waves by dropping the ion contribution as unimportant for motion on such fast time scales. Here, we will relax that assumption and consider lower frequency waves. (Section 4.2.1 of Bellan may be useful.)

   (a) Repeat the analysis done in class, but keep the contribution of the ions. In other words, linearize the electrostatic two-fluid equations for an initially uniform stationary neutral plasma. Show that the dispersion relation is given by the roots of the dielectric function given by

   $$\epsilon(\mathbf{k},\omega) = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - k^2 v_{th,s}^2},$$

   where $\omega_{ps} = 4\pi n_0 q_s^2/m_s$ is the plasma frequency of the species and $v_{th,s}^2 = \gamma_s T_{s0}/m_s$ is the thermal speed of the species. (Careful, the latter is non-standard notation!) Note, when the phase velocity of the waves $v_{ph} = \omega/k$ is comparable to $v_{th,s}$, the above result breaks down (as anticipated in Problem 1).

(b) To check our work from class, consider the very high frequency limit with $v_{ph} = \omega/k \gg v_{th,e,}, v_{th,i}$, *i.e.,* both electrons and ions are adiabatic ($\gamma_s = 3$). Show, by making reasonable assumptions and using the method of successive approximations, that the dispersion relation is given by

$$\omega^2 = \omega_{pe}^2 + k^2 v_{th,e}^2.$$

This, indeed, is the result we derived in class for the electron plasma waves when we neglected ion motion from the outset.

(c) Consider lower frequency waves which satisfy $v_{th,i} \ll \omega/k \ll v_{th,e}$, *i.e.,* the ions are adiabatic ($\gamma_i = 3$) but the electrons are isothermal ($\gamma_e = 1$). Show that the dispersion relation is approximately given by

$$\omega^2 = \frac{k^2 c_s^2}{1 + k^2 \lambda_{De}^2} + k^2 v_{th,i}^2$$

where $\lambda_{De} = T_{e0}/4\pi n_0 e^2$ is the electron Debye length and we define the "ion acoustic speed" $c_s^2 = T_{e0}/m_i$, the thermal speed based on the *ion* mass but the *electron* temperature. These waves are called ion acoustic waves.

(d) Consider the $T_{i0} \to 0$ limit. One might expect in this limit that there would be no sound waves of any form for the ions. However, from the dispersion relation we just derived, we discovered that there are! Let's figure out the physics of these waves, especially how the ions oscillate despite having no temperature. We have found that in the limit we are taking, the mass of the electrons is playing no role. Using the equations of motion for the electrons and ions in the $T_{i0}, m_e \to 0$ limits, explain the physics of ion acoustic waves.

(e) Again in the $T_{i0} \to 0$ limit, plot ω as a function of k for these waves, carefully labeling the plot. Consider using clever normalization to make your plot easier to read. What wave does the ion acoustic wave become for very small wavelength (large k) modes?