Phys 783 - Plasma Kinetic Theory
Assignment # 6
Due Friday, March 13, 2009

Reminders: Show your work! Include references on your submitted version. Write legibly!

1. **Energetics of Electron Plasma Waves**

   In class, we showed from kinetic theory that the energy density w of an electrostatic wave is

   $$w = \frac{\partial(\epsilon_r \omega_r)}{\partial \omega_r} \frac{|\hat{E}|^2}{8\pi}.$$

   Let's check this relation against fluid theory for the simplest waves we can find, the electron plasma waves with the ions forming a cold, stationary, charge neutralizing background species.

   (a) Recall that the real part of the dielectric function for electron plasma waves is

   $$\epsilon_r = 1 - \frac{\omega_{pe}^2}{\omega_r^2}.$$

   Calculate the energy density of these waves using the above expression for w.

   (b) Show the result same result comes from the fluid theory by finding the total energy invariant of the system. Specifically, start from the linearized fluid equations which govern electron plasma waves. Then, multiply the equation of motion by the velocity perturbation and Ampere's Law by the electric field perturbation and find a quantity whose time derivative is zero. Then, evaluate that energy invariant in terms of the electric field using the equation of motion and the dispersion relation to confirm that you get the same result.

   Note, you may have been able to jump right to the last step for this problem, but waves are not always this simple! The process developed here is useful for more complicated waves in which the answer might not be so easy to obtain.

2. **The Plasma Dispersion Function**

In class, we defined the plasma dispersion function as

$$Z(\xi) = \pi^{-1/2} \int_{-\infty}^{\infty} d\zeta \frac{1}{\zeta - \xi} e^{-\zeta^2}$$

(a) Show the following asymptotic expansions are valid in the limit of small imaginary part. For $\xi \gg 1$,

$$Z(\xi) = -\frac{1}{\xi}\left[1 + \frac{1}{2\xi^2} + \frac{3}{4\xi^4} + \cdots\right] + i\sqrt{\pi}e^{-\xi^2}$$

For $\xi \ll 1$,

$$Z(\xi) = -2\xi\left[1 - \frac{2\xi^2}{3} + \cdots\right] + i\sqrt{\pi}e^{-\xi^2}$$

Note - You should try it on your own first, but Section 5.3.5 of Bellan will be helpful if you get stuck.

(b) Starting from the integral expression, show that $Z(\xi)$ satisfies the differential equation

$$Z'(\xi) = -2[1 + \xi Z(\xi)],$$

where $Z' = dZ(\xi)/d\xi$.

(c) Show

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\zeta \frac{\zeta^2}{\zeta - \xi} e^{-\zeta^2} = -\frac{\xi}{2} Z'(\xi).$$

This integral arises in the derivation of the dielectric function for waves in a hot magnetized plasma.