Phys 783 - Plasma Kinetic Theory
Assignment # 7
Due Monday, March 30, 2009

Reminders: Show your work! Include references on your submitted version. Write legibly!

1. **Kinetic Theory of Electrostatic Ion Cyclotron Waves**

   In class, we discussed electrostatic waves in an unmagnetized plasma ($\mathbf{B}_0 = 0, \mathbf{B}_1 = 0$) and electromagnetic waves in a magnetized plasma ($\mathbf{B}_0 \neq 0, \mathbf{B}_1 \neq 0$). Here, we consider electrostatic waves in a magnetized plasma ($\mathbf{B}_0 \neq 0, \mathbf{B}_1 = 0$). The general dielectric function for electrostatic waves in a magnetized plasma (assuming an initially Maxwellian distribution function) is

$$\epsilon(\mathbf{k}, \omega) = 1 + \sum_s \frac{1}{k^2 \lambda_{Ds}^2} \left[1 + \left(\frac{\omega}{k_{||} v_{th,s}} \right) e^{-b_s} \sum_{n=-\infty}^{\infty} I_n(b_s) Z \left(\frac{\omega + n\Omega_{cs}}{k_{||} v_{th,s}} \right) \right]$$

   where $b_s = k_\perp^2 \rho_{Ls}^2/2 = k_\perp^2 v_{th,s}^2 / 2\Omega_{cs}^2$ is the normalized Larmor radius, I_n is the nth order modified Bessel function, $Z(\xi)$ is the Plasma Dispersion Function, and $v_{th,s}^2 = 2T_{s0}/m_s$ is the species thermal speed. This is Eq. (8.29) in Bellan (in different notation), and the interested reader will find its derivation preceding it.

   In this problem, we will find the dispersion relation for so-called "electrostatic ion cyclotron waves" or "EIC waves." They have been studied extensively by WVUs own Dr. Koepke!

   (a) Consider the limit in which $k_\perp^2 \rho_{Li}^2/2 \ll 1 \ll k_\perp^2 \rho_{Le}^2/2$, or equivalently, $b_i \ll 1 \ll b_e$ (meaning that the perpendicular wavelength is larger than the ion Larmor radius but smaller than the electron Larmor radius). Calculate the dielectric function to low order in b_e but first order in b_i. You may find the asymptotic relationships $I_n(x) \rightarrow e^x/(2\pi x)^{1/2}$ for large x and $I_n(x) \rightarrow (x/2)^n/n!$ for small x useful. Then, consider small temperatures so that $\omega + n\Omega_{cs} \gg k_{||} v_{th,s}$ and expand the Plasma Dispersion Function to low order in $v_{th,s}$. As a check, it is possible to write an intermediate step as

$$\epsilon \simeq 1 + \frac{1}{k^2 \lambda_{De}^2} + \frac{1}{k^2 \lambda_{Di}^2} \left\{ 1 - e^{-b_i} \left[1 + \frac{b_i}{2} \left(\frac{2\omega^2}{\omega^2 - \Omega_{ci}^2} \right) \right] \right\}.$$

   Since you are only keeping to first order in b_i, show that

$$\epsilon \simeq 1 + \frac{1}{k^2 \lambda_{De}^2} - \frac{k_\perp^2 v_{th,i}^2}{2k^2 \lambda_{Di}^2} \left(\frac{1}{\omega^2 - \Omega_{ci}^2} \right).$$

   Finally, set $\epsilon = 0$ and show that the dispersion relation for waves with $\omega^2 \simeq \Omega_{ci}^2$ is

$$\omega^2 = \Omega_{ci}^2 + k_\perp^2 c_s^2,$$

   where $c_s^2 = T_{e0}/m_i$ is the ion acoustic speed. This is the dispersion relation for electrostatic ion cyclotron waves, also obtainable from fluid theory.

   (b) Comment on the impact of Landau and cyclotron damping. In particular, are the waves heavily damped or negligibly damped?

2. **Kinetic Theory of Parallel Propagating Waves**

In class, we showed that parallel propagating waves ($k_\perp = 0$) in a magnetized plasma (with a Maxwellian equilibrium) satisfy $\epsilon_{xx} = \pm i\epsilon_{xy}$. Here, we investigate the fluid limit of these waves and their damping. The relevant elements of the dielectric tensor are

$$\begin{aligned} \epsilon_{xx} &= 1 - \frac{k_{||}^2 c^2}{\omega^2} + \sum_s \frac{\omega_{ps}^2 e^{-b_s}}{\omega k_{||} v_{th,s} b_s} \sum_{n=-\infty}^{\infty} n^2 I_n(b_s) Z(\xi_{sn}) \\ \epsilon_{xy} &= i \sum_s \frac{s\omega_{ps}^2 e^{-b_s}}{\omega k_{||} v_{th,s}} \sum_{n=-\infty}^{\infty} n[I_n(b_s) - I_n'(b_s)] Z(\xi_{sn}) \end{aligned}$$

where $\xi_{sn} = (\omega + n\Omega_{cs})/k_{||}v_{th,s}$, and the $\pm$ sign in the notes in the ϵ_{xy} term is replaced by s where $s = 1$ for ions and -1 for electrons to avoid confusion with the other $\pm$ symbol.

(a) Since $b_s \propto k_\perp^2$, find the $b_s \to 0$ limit of ϵ_{xx} and ϵ_{xy}. Treat the small temperature limit by keeping only the lowest order term in the expansion of $Z(\xi_{sn})$ for large ξ_{sn}. Then, show that setting $\epsilon_{xx} = \pm i\epsilon_{xy}$ implies a dispersion relation of

$$\frac{k_{||}^2 c^2}{\omega^2} = 1 - \sum_s \frac{\omega_{ps}^2}{\omega} \frac{1}{\omega \pm s\Omega_{cs}}.$$

These are the R (plus sign) and L (minus sign) waves from fluid theory [compare Bellan, Eq. (6.28)].

(b) Consider the R wave and solve for ω in the limit of $\Omega_{ci}^2 \ll \omega^2 \ll \Omega_{ce}^2$. Show that

$$\omega \simeq k_{||}^2 c^2 \Omega_{ce}/\omega_{pe}^2.$$

[Hint - Take the low order terms for each species, and show that the ion term is small compared to the electron term. You could also show (*but you don't have to*) that the electron term dominates the "1" term if $\omega_{pe}^2 \gg \Omega_{ce}^2$ (a reasonable assumption for most plasmas). Then solve for ω.] This is the "whistler" wave, which is very important to Dr. Scime's helicon research and Dr. Cassak's reconnection research.

(c) Now consider low frequency waves, such that $\omega^2 \ll \Omega_{ci}^2, \Omega_{ce}^2$. In this limit, the coefficient of the sum on n in ϵ_{xy} becomes large. As such, one of the two solutions is when $\epsilon_{xy} = 0$. The other is when $\epsilon_{xx} = 0$. Using your expression for ϵ_{xx} from part (a), take the $\omega^2 \ll \Omega_{ci}^2, \Omega_{ce}^2$ limit, finding a dispersion relation of

$$\omega^2 \simeq k_{||}^2 c_{Ai}^2,$$

where $c_{Ai}^2 = B_0^2/4\pi m_i n_{i0}$ is the ion Alfvén speed. [Hint - Again take the low order terms, but now show that the ion term dominates the electron term. (It dominates the "1" too.)] This is the dispersion relation for Alfvén waves, a fundamental low-frequency large-wavelength wave which arises in magnetohydrodynamics.

(d) Finally, consider damping of Alfvén waves. Keep the imaginary part of the Plasma Dispersion Function and show that the damping rate ω_i for Alfvén waves is

$$\omega_{i,\text{Alfven}} = -\frac{\sqrt{\pi}}{2} \frac{\omega_{pi}^2}{k_{||} v_{th,i}} \frac{c_{Ai}^2}{c^2} e^{-\Omega_{ci}^2/k_{||}^2 v_{th,i}^2}$$

For **extra credit**, show that the damping rate for whistler waves is

$$\omega_{i,\text{whistler}} = -\sqrt{\pi} \left(\frac{\omega_{pe}^2}{k_{||} v_{th,e}} \right) \left(\frac{k_{||}^2 c^2 \Omega_{ce}^2}{\omega_{pe}^4} \right) e^{-\Omega_{ce}^2/k_{||}^2 v_{th,e}^2}$$

Damping for both waves is weak, justifying a fluid description.