

Reminders: Show your work! Include references on your submitted version. Write legibly!

### 1. “Universal” Drift Wave Dispersion Relation from Vlasov Theory

The full dielectric function for electrostatic drift waves, as discussed in class, is

$$\epsilon(\mathbf{k}, \omega) = 1 + \sum_s \frac{1}{k^2 \lambda_{Ds}^2} \left[ 1 + \left( \frac{\omega - \omega_{*s}}{k_{\parallel} v_{th,s}} \right) e^{-b_s} \sum_{n=-\infty}^{\infty} I_n(b_s) Z(\xi_{sn}) \right], \quad (1)$$

where  $b_s = k_{\perp}^2 \rho_{Ls}^2 / 2$ ,  $\xi_{sn} = (\omega + n\Omega_{cs}) / k_{\parallel} v_{th,s}$ , and  $\omega_{*s} = cT_{s0} / q_s L_n B_0$ . The parameter regime which leads to universal drift waves is

$$\frac{1}{2} k_{\perp}^2 \rho_{Li}^2, \frac{1}{2} k_{\perp}^2 \rho_{Le}^2 \ll 1 \quad (2)$$

$$k_{\parallel} v_{th,i} \ll \omega \ll k_{\parallel} v_{th,e} \quad (3)$$

$$\omega^2 \ll \Omega_{ci}^2, \Omega_{ce}^2 \quad (4)$$

(a) Show that  $\epsilon$  in the above limit can be written as

$$\epsilon(\mathbf{k}, \omega) = 1 + \frac{1}{k^2 \lambda_{Di}^2} \left[ 1 - e^{-b_i} I_0(b_i) + \left( \frac{\omega_{*i}}{\omega} \right) e^{-b_i} I_0(b_i) \right] + \frac{1}{k^2 \lambda_{De}^2} + \frac{i\sqrt{\pi}}{k^2 \lambda_{De}^2} \left( \frac{\omega - \omega_{*e}}{k_{\parallel} v_{th,e}} \right).$$

To do this, you can follow these steps: (1) In the ion term, Eq. 2 implies  $b_i \ll 1$ , so keep only the zeroth and first order terms in  $b_i$ . (Be careful!) Then, use Eq. 3 to expand  $Z$  in the appropriate limit. Use Eq. 4 to simplify. (2) In the electron term, Eq. 2 implies  $b_e \ll 1$ , so expand for small  $b_e$ , keeping only to zeroth order for simplicity. Then, use Eq. 3 to expand  $Z$  in the appropriate limit. (3) Put the ion and electron terms back together, and drop negligible terms. **Hint** - Pages 302-3 of Bellan may be useful as a guide.

(b) Show that the real part of the frequency  $\omega_r$  is

$$\omega_r = \omega_{*e} \left( \frac{e^{-b_i} I_0(b_i)}{1 + (T_{e0}/T_{i0})[1 - e^{-b_i} I_0(b_i)]} \right)$$

by setting  $\epsilon_r = 0$  as usual. Near the end, also take the  $k^2 \lambda_{De}^2 \ll 1$  limit. Next, expand this result to first order in  $b_i$  to show that the result can be written as

$$\omega_r = \omega_{*e} \left( \frac{1 - b_i}{1 + T_{e0} b_i / T_{i0}} \right) = \omega_{*e} \left( \frac{1 - k_{\perp}^2 \rho_{Li}^2 / 2}{1 + k_{\perp}^2 \rho_s^2} \right),$$

where  $\rho_{Li}^2 = v_{th,i}^2 / \Omega_{ci}^2$  as usual, and we have defined  $\rho_s^2 = c_s^2 / \Omega_{ci}^2$  as the ion Larmor radius based on the ion acoustic speed  $c_s^2 = T_{e0} / m_i$ . (Yes, the notation is confusing.)

(c) Using the form in which you have already expanded to first order in  $b_i$ , use standard techniques from class to show that the imaginary part of the frequency  $\omega_i$  is

$$\omega_i = \sqrt{\pi} \frac{\omega_{*e}^2}{k_{\parallel} v_{th,e}} \frac{b_i + k_{\perp}^2 \rho_s^2}{(1 + k_{\perp}^2 \rho_s^2)^3} = \sqrt{\pi} \frac{\omega_{*e}^2}{k_{\parallel} v_{th,e}} \frac{k_{\perp}^2 (\rho_{Li}^2 / 2 + \rho_s^2)}{(1 + k_{\perp}^2 \rho_s^2)^3}.$$

As noted in class,  $\omega_i$  is positive definite, so the wave is unstable.

(d) Briefly, what physics drives the universal drift wave unstable? (You can get this by looking at the part of  $\epsilon$  that gives rise to the imaginary part of  $\omega$ .)

## 2. The Lower Hybrid Drift Instability

We now consider the same dielectric function but in a different parameter regime, which will give us the important “lower hybrid drift instability” (LHDI). The parameter regime which leads to the LHDI is

$$\frac{1}{2}k_{\perp}^2\rho_{Li}^2, \frac{1}{2}k_{\perp}^2\rho_{Le}^2 \ll 1 \quad (5)$$

$$\omega \ll k_{\parallel}v_{th,i}, k_{\parallel}v_{th,e} \quad (6)$$

$$\Omega_{ci}^2 \ll \omega^2 \ll \Omega_{ce}^2 \quad (7)$$

(a) Show that  $\epsilon$  in the above limit can be written as

$$\epsilon(\mathbf{k}, \omega) = 1 + \frac{1}{k^2\lambda_{De}^2} \left[ 1 - e^{-b_e} I_0(b_e) + \left( \frac{\omega_{*e}}{\omega} \right) e^{-b_e} I_0(b_e) \right] + \frac{1}{k^2\lambda_{Di}^2} + \frac{i\sqrt{\pi}}{k^2\lambda_{Di}^2} \left( \frac{\omega - \omega_{*i}}{k_{\parallel}v_{th,i}} \right).$$

Note - the result looks like that in the previous problem but with the ion and electron contributions interchanged, even though the physics is very different! To derive the above expression, you can follow this procedure: (1) For the ion term, keep only to first order in  $b_i$  using Eq. 5. Then, using Eqs. 7 and 6, expand  $Z$  and keep the lowest order terms necessary, arguing that the  $n = \pm 1$  are negligible. (2) Perform the required expansion for the electron term. You are encouraged to use your result from Problem 1 to avoid redoing the algebra.

(b) In a similar fashion to Problem 1, show that the real part of the frequency  $\omega_r$  is

$$\omega_r = -\omega_{*e} \left( \frac{e^{-b_e} I_0(b_e)}{1 + T_{e0}/T_{i0} - e^{-b_e} I_0(b_e)} \right),$$

or, to first order in  $b_e$ ,

$$\omega_r = \omega_{*i} \left( \frac{1 - b_e}{1 + T_{i0}b_e/T_{e0}} \right) = \omega_{*i} \left( \frac{1 - k_{\perp}^2\rho_{Le}^2/2}{1 + (T_{i0}/T_{e0})k_{\perp}^2\rho_{Le}^2/2} \right).$$

(c) As in Problem 1, use the form in which you have already expanded to first order in  $b_e$ , to show that the imaginary part of the frequency  $\omega_i$  is

$$\omega_i = \sqrt{\pi} \frac{\omega_{*i}^2}{k_{\parallel}v_{th,i}} \frac{b_e(1 + T_{i0}/T_{e0})}{(1 + T_{i0}b_e/T_{e0})^3} = \sqrt{\pi} \frac{\omega_{*i}^2}{k_{\parallel}v_{th,i}} \frac{k_{\perp}^2\rho_{Le}^2}{2} \frac{1 + T_{i0}/T_{e0}}{(1 + k_{\perp}^2\rho_{Le}^2 T_{i0}/2T_{e0})^3}$$

As you might have expected, this reveals that  $\omega_i$  is positive definite, implying that this wave is unstable.

(d) Briefly, what physics drives the LHDI unstable? (Again, consider the part of  $\epsilon$  that gives rise to the imaginary part of  $\omega$ .)

Comments - From Eq. 7, the LHDI occurs for frequencies intermediately between  $\Omega_{ci}$  and  $\Omega_{ce}$ . This is reminiscent of a type of fluid waves with a frequency called the lower hybrid frequency  $\omega_{lh} = (\Omega_{ci}\Omega_{ce})^{1/2}$ , which also satisfies the same frequency criterion. (See Bellan, Section 6.3.3 for more information.) This helps to motivate the name of the instability. Also note that attempting to give a fluid theory of the LHDI does not successfully reproduce the kinetic theory result, so this is an example where kinetic theory is necessary to get the physics right!