Dynamics of falling raindrops

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Abstract
A standard undergraduate mechanics problem involves a raindrop which grows in size as it falls through a mist of suspended water droplets. Ignoring air drag, the asymptotic drop acceleration is \( g/7 \), independent of the mist density and the drop radius. Here we show that air drag overwhelms mist drag, producing drop accelerations of order \( 10^{-3} g \). Analytical solutions are facilitated by a new empirical form of the air drag coefficient \( C = 12R^{-1/2} \), which agrees with experimental data on liquid drops in the Reynolds-number range \( 10 < R < 1000 \) relevant to precipitating spherical drops. Solutions including air drag are within reach of students of intermediate mechanics and nonlinear dynamics.

Even without air drag, the dynamics of a raindrop falling through a stationary mist serves as an important and non-trivial application of Newton’s second law because the mass of the drop changes with time. Undergraduate mechanics students are sometimes able to solve the nonlinear dynamical equations of motion to find the deceptively simple acceleration \( g/7 \) of an infinitesimal-radius drop released from rest, assuming that the drop accretes all the mist that it encounters. Dick [1] showed that drops of arbitrary initial radius and velocity approach this acceleration asymptotically. Krane [2] confirmed that inelastic collisions account for the lost mechanical energy of the falling drop. Partovi and Aston [3] included air drag in the problem, assuming a constant drag coefficient for pedagogical simplicity.

The objective of this paper is to include the variations in the air drag coefficient for growing raindrops. As raindrops grow in radius from \( r = 0.1 \text{ mm} \) to \( r = 1 \text{ mm} \) within a cloud, their drag coefficients decrease from about \( C = 5 \) to about \( C = 0.5 \). To account for this decrease, we employ a simple but accurate empirical relationship for the dependence of the drag coefficient on the Reynolds number, which allows us to obtain simple exponential solutions for the asymptotic drop radius, speed, acceleration, and distance travelled. Because of their accuracy, these solutions closely mimic the behaviour of real raindrops, and predict the actual time required for a raindrop to fall through a cloud. Because of their simplicity, these solutions are accessible to students of intermediate mechanics and nonlinear dynamics, who benefit by this soluble yet realistic example.

Our approach to the problem is couched in the language and formalism of modern nonlinear dynamics. Since air densities \( \rho_a \approx 10^{-3} \text{ g cm}^{-3} \) greatly exceed the mist densities [4, 5] \( \rho_m \approx 10^{-6} \text{ g cm}^{-3} \), air drag might be expected to play an important role in raindrop dynamics. Air drag indeed overwhelms the force of the accreting mist
A spherical raindrop of mass $m$, radius $r$, vertical velocity $v$ (positive downward), and mass density $\rho_d \approx 1$ g cm$^{-3}$ satisfies

\[ m = \frac{4}{3} \pi \rho_d r^3 \]  

\[ \frac{dm}{dt} = \pi \rho_m r^2 |v| \]  

\[ m \frac{dv}{dt} = mg - \frac{C}{2} \pi \rho_d r^2 v |v| - \frac{dm}{dt} \]  

where the terms on the right-hand side of equation (3) give the weight, air drag, and mist drag acting on the growing raindrop.

When the air drag is written as above, the drag coefficient $C = C(R)$ is a function only of the Reynolds number $R = 2r |v| / \nu$, where $\nu$ is the kinematic viscosity of air. Figure 1 shows $C(R)$ measured for falling liquid drops (data points, [8]), together with the small-$R$ theoretical Stokes result $C = 24/R$ (broken line, [6]) and a new empirical result (full line), $C = 12R^{-1/2}$ (4)

valid over the intermediate range $10 < R < 1000$. Such simple algebraic results allow us to avoid complicated numerical results for $C(R)$ obtained from the nonlinear Navier–Stokes equations [7]. Inserting equation (4) into equation (3) and setting $\frac{dm}{dt} = \frac{dv}{dt} = 0$ yields the mist-free terminal or ‘settling’ speed of drops of radius $r$ with $10 < R < 1000$:

\[ v_0 = \sigma r \]
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Figure 2. Predicted (full line) and experimental (data points) mist-free settling speeds $v$ of raindrops as a function of radius $r$, together with values of the Reynolds number $R = 2 vr / \nu$, for kinematic viscosity $\nu = 0.2064 \text{ cm}^2 \text{s}^{-1}$ and falling rate $\sigma = 8560 \text{ s}^{-1}$ relevant to warm clouds at an elevation of 3 km, pressure of 700 mb, and a temperature 10 °C.

$$v (\text{m/s})$$

$$r (\text{mm})$$

where

$$\sigma = \frac{2}{3} \left( \frac{\rho_d}{\rho_a} \right)^{2/3} \left( \frac{g^2}{3 \nu} \right)^{1/3}$$

Figure 2 compares measured mist-free settling speeds (data points, [8]) with the prediction of equations (5) and (6) (full line), for the value $\sigma = 8560 \text{ s}^{-1}$ pertinent for a typical warm rain cloud of an elevation of 3 km, pressure of 700 mb, and a temperature of 10 °C, for which [7, 9] $\rho_a = 0.856 \times 10^{-3} \text{ g cm}^{-3}$, $\rho_d = 1 \text{ g cm}^{-3}$, $\nu = 0.206 \text{ cm}^2 \text{s}^{-1}$, and $g = 980 \text{ cm s}^{-2}$. Raindrops with $R > 10$ and $r > 0.1 \text{ mm}$ have settling velocities, $v_0 > 1 \text{ m s}^{-1}$, large enough to be designated as ‘precipitating’ drops [8]. Smaller ‘cloud’ droplets follow the convection currents in the cloud much more closely, with the smallest 1 μm aerosol droplets being fully entrained in the flow, and being subject to significant accretion by Brownian motion [8], ignored here. Large raindrops with $R > 1000$ and $r > 1 \text{ mm}$ flatten because surface tension, which maintains the sphericity of smaller drops, succumbs to the effects of the air flow on the particle shape [5]. Raindrops with equivalent spherical radii $r > 4.5 \text{ mm}$ break up spontaneously because the drag stress exceeds the surface tension stress [7]. We accordingly confine our attention to spherical precipitating drops with $10 < R < 1000$ and $0.1 \text{ mm} < r < 1 \text{ mm}$. Thin clouds release such small drops, called drizzle.

The associated nonlinear dynamical equations of motion

$$\frac{dv}{dt} = g \left[ 1 - \frac{v |v|^{1/2}}{(\sigma r)^{3/2}} \right] - 3 \epsilon \frac{v |v|}{r}$$

(7)
follow by inserting equations (1), (2), and (4) into equation (3), and by inserting equation (1) into equation (2), with $\varepsilon = \rho_m/4\rho_d$. To study $r-v$ phase plane trajectories, it is useful to eliminate time by dividing (7) by (8), and by treating $v$ as a function of $r$. Specializing to falling drops with $v > 0$, we obtain

$$\frac{d}{dt} v^2 = g \left[ 1 - \left(\frac{v}{\sigma r}ight)^{3/2} \right] - 3\varepsilon \frac{v^2}{r}. $$

For no air drag ($\sigma \to \infty$), equation (9) easily reduces to

$$r \frac{d}{dr} v^2 = \frac{2g}{\varepsilon} \frac{v^2}{r}. $$

Accordingly, since $r$ increases monotonically with time, arbitrary $r-v$ phase-space trajectories asymptotically approach an attractor satisfying $v^2/r = 2g/7\varepsilon$, for which the right-hand side of equation (10) vanishes. Substituting this result and $\sigma \to \infty$ into (7) immediately gives the classical drag-free acceleration $\frac{dv}{dt} = g/7$. An infinitesimal-radius drop released from rest therefore traces out the entire attractor, according to the power-law time dependences $v = (g/7)t$ and $r = (\varepsilon g/14)t^2$.

We now consider both mist and air drag. Since clouds rarely exceed the values $\rho_m = 1 \times 10^{-6}$ g cm$^{-3}$ and $\varepsilon = 2.5 \times 10^{-7}$, the role of mist drag is small compared with air drag, and we can expand according to

$$v(r) = v_0(r) + \varepsilon v_1(r) + \cdots. $$

Expanding equation (9) accordingly gives the zeroth- and first-order results $v_0 = \sigma r$ (equation (5), the result for no mist drag) and $v_1 = -(8/3g)\sigma^3 r^2$, whence

$$v(r) = \sigma r \left( 1 - \frac{8\sigma^2}{3g} r \right)$$

valid through first order in $\varepsilon$. This equation predicts the speed approached asymptotically by all $r-v$ phase-space trajectories. Accordingly, a drop whose speed and radius do not obey equation (12) will initially adjust both until they do. Mist and air drag dominate over the drop weight for falling drops whose speeds exceed those given by (12), causing the drops to slow down, while the weight dominates for speeds smaller than those given by (12), causing the drops to speed up. Thus equation (12) represents an attractor in phase space, to which arbitrary phase-space trajectories are drawn asymptotically. Equation (11) precludes transient trajectories off the attractor because such trajectories satisfy $r \to$ constant and $\frac{dv}{dr} \to \pm \infty$ as $\varepsilon \to 0$. The small 0.5% contribution of the first-order term in (12) confirms the validity of the expansion on the attractor, and implies that the speeds predicted by equation (12) are barely smaller than their mist-free settling speeds. With mist, however, the drop speeds increase as the drop radius increases with time because the settling speed of a drop is proportional to its radius. Ignoring the small first-order correction in equation (12) and applying equation (8) yields the lowest-order exponential time dependences for raindrops whose trajectories lie on the attractor,

$$r = r_0 e^{\sigma t} $$

$$v = \sigma r_0 e^{\sigma t} $$

$$a = \varepsilon \sigma^2 r_0 e^{\sigma t} $$

$$z = \frac{r_0}{\varepsilon} e^{\sigma t} $$

where $r_0$ is the initial drop radius, $a = dv/dt$ is the acceleration of the drop, and $z$ is the distance fallen in time $t$, satisfying $v = dz/dt$. In place of these exponential dependences, Partovi and Aston [3] obtained polynomial time dependences for spherical raindrops with a
constant drag coefficient. They also considered non-spherical raindrops, relevant for raindrop radii larger than those considered here.

Figure 3 shows phase trajectories resulting from numerical integrations of equations (7) and (8) for $\sigma = 8560 \text{ s}^{-1}$ and $g = 980 \text{ cm s}^{-2}$. These integrations were carried out using a fourth-order Runge–Kutta method with adaptive step size. For the physical value $\epsilon = 2.5 \times 10^{-7}$ (long-dashed curves), raindrops quickly approach the attractor given by (12) (trace A) before their radii change appreciably. To help to clarify the dynamics, we also include numerical trajectories for the larger unphysical value $\epsilon = 1 \times 10^{-5}$ (short-dashed curves) associated with unphysically large mist densities, for which some change in radius is perceptible as the trajectories approach the numerically-calculated attractor (trace B), which differs slightly from the prediction of equation (12) (trace $B'$) at large $r$. For the physical value $\epsilon = 2.5 \times 10^{-7}$, the numerically-calculated attractor is indistinguishable from the prediction of (12); both lie on trace A.

It is instructive to study the life history of a typical raindrop of radius $r_0 = 0.1 \text{ mm}$ which is released from rest, and which grows until its radius reaches $r = 1 \text{ mm}$. For $\epsilon = 2.5 \times 10^{-7}$, this drop follows trajectory C in figure 3 as it approaches the attractor, and closely follows trajectory A, the attractor, thereafter. Although the drop trajectory never actually reaches the attractor, the drop speed comes to within a fraction, $1 - \delta$, of the speed on the attractor within a time

$$\tau_1 = \frac{\sigma r_0}{3g} \left( \ln \frac{3}{\sqrt{2}} - \frac{\pi}{\sqrt{3}} \right)$$

This time is obtained by integrating equation (7) with $\epsilon = 0$ and $r = r_0$, and is valid to leading order in $\delta$ and for small $\epsilon$. Equation (7) implies an acceleration which begins at $a = g$ when the
drop is released from rest, and decreases quickly to the value \( a = 1.87 \times 10^{-4} \text{g} \) (equation (15)) as the drop reaches the attractor. The raindrop then spends a time (equation (13))

\[
\tau_2 = \frac{1}{\varepsilon \sigma} \ln \frac{r}{r_0}
\]

\( \tau_2 \) is the time on the attractor as it grows to its final radius. For \( \delta = 0.0001 \), equations (17) and (18) give \( \tau_1 = 0.516 \text{ s} \) and \( \tau_2 = 1076 \text{ s} \), which agree with values from our numerical simulations to within 0.1%. Clearly, the half-second spent initially by the drop in reaching its settling speed is small compared with the almost 18 min spent slowly growing and accelerating in the cloud. The ratio \( \tau_1/\tau_2 \) scales as \( \varepsilon \sigma^2 r_0/3g \), just as the first-order term in equation (12), and is insensitive to the specific choice of \( \delta \). The final velocity \( v = 8.56 \text{ m s}^{-1} \), acceleration \( a = 1.87 \times 10^{-3} \text{g} \), and distance fallen \( z = 4000 \text{ m} \) follow by substituting \( t = \tau_2 \) into equations (14)–(16). For comparison, recreational runners routinely run 4000 m in 18 min, and competitive sprinters routinely beat the final velocity of \( v = 8.56 \text{ m s}^{-1} \). The final acceleration \( a = 1.87 \times 10^{-3} \text{g} \) is tiny compared with the value \( g/7 \) for the drag-free raindrop problem. The distance fallen is typical of the thickness of cumulus clouds [5].

In conclusion, one might expect that mist drag would dominate over air drag for falling raindrops because condensed droplets are massive compared to air molecules. In contrast to this expectation, air drag dominates; spherical raindrops fall at speeds only slightly less than their mist-free settling speeds, apart from short transients needed for them to reach these speeds. The reason is that the mass density of air is a thousand times larger than the mass density of mist in terrestrial rain clouds. Consequently, the inertia of the air dominates over that of the mist. Mist drag increases in importance relative to air drag with increasing drop radius because the air drag coefficient decreases with increasing Reynolds number. Even so, spontaneous raindrop breakup for \( r > 4.5 \text{ mm} \) precludes the observation of any appreciable effect.

In spite of these arguments, the mist is critical to understanding the dynamics of raindrops because it slowly increases the raindrop radius and settling speed. Thus, raindrops never fully reach a ‘terminal’ velocity, but continue to accelerate slowly as they grow, with accelerations \( a \approx 10^{-3} \text{g} \) which are much smaller than the unobservable drag-free value \( a = g/7 \).

References