Experimental verification of periodic pulling in a nonlinear electronic oscillator

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An experimental investigation of entrainment and the phenomenon known as periodic pulling is described. Periodic pulling refers to the incomplete entrainment of an oscillatory nonlinear system by a periodically varying driving force. The process whereby the system's oscillation frequency is pulled toward the driving frequency stops short of complete synchronization and is interrupted at regular intervals. In this way, periodic pulling produces pulselike amplitude and frequency modulation in the system's oscillatory response. Associated with these combined, nonsinusoidal modulations is an asymmetric, or single-sided, spectral peak made up of spectral components at frequencies incommensurate with the undriven (spontaneous) and driving frequencies. The system being investigated uses a unijunction transistor as the nonlinear element in an electrical circuit. The behavior of the system is shown to resemble that predicted for a forced van der Pol oscillator with an adjustable nonlinear restoring force.

For certain ranges of driving frequency and amplitude, when the system is not entrained to the driving force, periodic pulling is shown to alter the system's oscillatory response significantly from that expected for conventional amplitude modulation involving two sinusoids. The essential features of the periodic pulling phenomenon are observed and compared, with good agreement, to a model.

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I. INTRODUCTION

An electrical RLC circuit (i.e., one with discrete elements of resistance, capacitance, and inductance connected together in series) in which the resistance has a negative characteristic may exhibit many of the nonlinear properties [1] associated with real-world oscillators [2]. Of these properties, the one most relevant to the work described in this paper is the existence of a stable limit cycle [3]. Studies by van der Pol [4] provided a quantitative model of nonlinear oscillatory behavior that has since been used extensively in the characterization and interpretation of self-oscillatory systems with small or large nonlinearity [5–7].

A particular feature of these systems when they are subject to a periodically varying driving force is the occurrence of entrainment [8], or synchronization [9–14], of the response of the system to the driving frequency. Typically, the amplitude of the driving force necessary for entrainment is very small compared to the amplitude of the spontaneous oscillation and varies directly with the frequency difference between the sinusoidal driving force and the spontaneous oscillation. For a specific value of driving amplitude, there is a range of driving frequencies for which the system is entrained. The ability to entrain a large-amplitude oscillator to a small-amplitude oscillator has been applied to electronic [15], mechanical [16], chemical [17], and biological [18] oscillators.

Outside the entraining range, a driving force affects the response of the system by producing a modulation of the oscillation amplitude. In conventional amplitude modulation [19], a pair of spectral features (sidebands) appears, centered around the spontaneous frequency in the oscillation spectrum. The two features are separated from each other by twice the frequency difference between the spontaneous frequency and the driving frequency. Ordinarily, a simple relation connects the size of these sidebands to the amplitude of the driving force and another simple relation connects the location of these sidebands to the frequency of the driving force. These simple relations do not describe the system's response if periodic pulling occurs.

Periodic pulling [20] refers to the incomplete entrainment [21,22] of an oscillatory nonlinear system by a periodically varying driving force. The process whereby the system's oscillation frequency gets pulled toward the driving frequency stops short of complete synchronization and is interrupted at regular intervals, at which point the oscillation frequency briefly unlocks from its pulled value. At the interruption points, the oscillation amplitude may change suddenly, very likely as a result of the phase mismatch that develops between the driver and the system's response during the interval. After such an interruption, the oscillation amplitude recovers and then decreases slowly.

The pulselike fashion of the frequency modulation (FM) and the amplitude modulation (AM) introduces a series of sidebands. As in any pulselike time series, the associated sidebands are spaced at frequency intervals identified with multiples of the pulsing frequency. A sideband cancellation effect [20,23], due to the phase relationship between sidebands produced by AM and FM, results in an asymmetric or single-sided spectral profile for the oscillations. The cancellation always takes place on the side of the driving frequency opposite the spontaneous frequency.

The experimental results reported in this paper are measurements made on a relatively simple nonlinear electronic circuit that uses a unijunction transistor (UJT), a three-terminal semiconductor device, as the nonlinear
element. The most common application of the UJT is the relaxation oscillator circuit [24]. With a different circuit, relatively unknown and seldom recognized periodic pulling behavior is detected in the system’s response. The excellent controllability and reproducibility of the experiments described here permit a more thorough experimental characterization of the phenomenon than was previously available in the literature. This in turn leads to a more thorough comparison between the observed behavior and that predicted theoretically. It is found that the observed behavior is well modeled by the theory.

A model of a forced van der Pol oscillator with nonlinear restoring force, as well as a theoretical description of periodic pulling, are reviewed in Sec. II to arrive at the predictions with which the experimental results are to be compared. In Sec. III, the experimental system is described and the experimental results are presented. Section IV contains a discussion of the results.

II. THEORY

The van der Pol equation [4]

\[ \ddot{x} - \epsilon (1 - \beta x^2) \omega_0 \dot{x} + \omega_0^2 x = 0 \]  \hspace{1cm} (1)

can be used to model self-oscillatory behavior because of the nonlinear dissipation term. For the spontaneous oscillation described by Eq. (1), the amplitude approaches a steady-state value from either smaller or larger initial values. The attractor for this system is said to be a stable limit cycle.

Consider a solution of Eq. (1) that is of the form [25]

\[ x = a(t) \sin[a(t) + \psi(t)] \]  \hspace{1cm} (2)

with arbitrary initial values \( a(0) \) and \( \psi(0) \). The steady-state values of \( a \) and \( \psi \) are reached asymptotically in a characteristic time \( 2\pi/\epsilon \omega_0 \). The (period-averaged) steady-state value of \( \psi \) is zero so that, on average, \( \psi \) is constant and the spontaneous frequency of self-oscillation can be considered to be \( \omega_0/2\pi \equiv f_0 \). The (period-averaged) steady-state value of \( a \) is (4/\( \beta \))\( \sqrt{2} \equiv a_0 \), where \( \beta \) is a measure of the nonlinearity of the system.

A sinusoidal forcing term can be added to the right-hand side of Eq. (1) to model the behavior of a self-oscillating system that is subjected to a periodically varying driving force. The so-called forced van der Pol equation is

\[ \ddot{x} - \epsilon(1 - \beta x^2) \omega_0 \dot{x} + \omega_0^2 x = M \omega_0^2 \sin(\omega_D t) \]  \hspace{1cm} (3)

where the driving-force frequency is \( \omega_D/2\pi \equiv f_D \) and the driving-force voltage is \( M \sin(\omega_D t) \). For a specific value of driving amplitude, there is a range of driving frequencies for which the response of the system is synchronized, or entrained, to the driving frequency.

For this system, the effect of a nonlinear restoring force can be modeled with an additional nonlinear in Eq. (3),

\[ \ddot{x} - \epsilon(1 - \beta x^2) \omega_0 x + (\omega_0^2 + \eta x^2) x = M \omega_0^2 \sin(\omega_D t) \]  \hspace{1cm} (4)

The solution to either Eq. (3) or (4) can be of the form

\[ x = a \cos(\omega_D t + \theta) + b \cos(\omega_D t + \phi) \]  \hspace{1cm} (5)

where \( a \) and \( b \) are the amplitudes of the spontaneous oscillation and the forced oscillation, respectively, and the values of \( \theta \) and \( \phi \) are constant. The steady-state behavior of this system when it is entrained (Condition 1) and when it is not entrained (Condition 2) can be described by the following relationships [Eqs. (6)–(10)], taken from Ref. [26].

**Condition 1.** Entrained (i.e., \( a = 0 \)):

\[ \rho^2(\rho^2 - 1)^2 + (J f^2 - \delta^2) = K \]  \hspace{1cm} (6)

**Condition 2.** Not entrained (i.e., \( a \neq 0 \)):

\[ \rho^2(3\rho^2 - 1)^2 + [J(2 - 3\rho^2) - \delta^2] = K \]  \hspace{1cm} (7)

where \( \omega_D - \omega_0 \) is small compared to either \( \omega_D \) or \( \omega_0 \) and where

\[ \delta = \frac{\omega_D - \omega_0}{\epsilon \omega_0}, \quad \rho = \frac{b}{a_0}, \quad J = \frac{3\eta}{\epsilon \omega_0^2 \beta}, \quad K = \frac{M}{\epsilon \omega_0^2} \]  \hspace{1cm} (8)

The so-called response curves of \( \rho \) versus \( \delta \) from Eqs. (6) and (7) represent the frequency dependence of the system’s oscillation amplitude, with \( K \) as a parameter. The value of amplitude maximum \( \rho_m \) for each curve depends on driving amplitude according to

\[ \rho_m^2(\rho_m^2 - 1)^2 = K \]  \hspace{1cm} (9)

and depends on driving frequency according to

\[ \rho_m^2 = \frac{\delta^2}{J} \]  \hspace{1cm} (10)

Between conditions 1 and 2 above, at the threshold of entrainment, the critical value of driving amplitude and the driving frequency are related by

\[ \frac{1}{4}(J - \delta^2) = K \]  \hspace{1cm} (11)

The forced van der Pol model has been used by some authors to interpret experimental observations of oscillations in systems that were classified as being either in or out of entrainment. Abrams, Yadlowsky, and Lashinsky [27] and Lashinsky, Rosenberg, and Detrick [28], however, reconized that not one, but two different classes of oscillations exist when the system is not entrained and obtained spectra from both classes in experiments involving plasma drift waves. One class, labeled “combination oscillations,” consists of components at frequencies that are combinations of \( f_0, f_D, \) and \( |f_D - f_0| \). The other class, labeled “almost-periodic oscillations,” occurs when the values of driving amplitude and frequency are close to the region of entrainment in parameter space. The transition from one class to the other as observed in the spectral content of the system’s response has been reported by Hakki, Beccione, and Plauski [29] in a bulk GaAs cw microwave oscillator, by Stover [30] in a tunnel-diode oscillator [6], and by Faitare, Peyrand, and Pointu [31] and Amemiya [32] in positive column discharges.

Hayashi [21] points out that the amplitude and phase of an almost-periodic oscillation varies slowly but periodically, even in the steady state. However, since the ratio between the period of the amplitude variation and that of the driving force is, in general, incommensurable, there is no periodicity in the almost-periodic oscillation. He
gives relations that determine the time-varying coefficients $c_1$ and $c_2$ in the expression for the system's oscillatory response,

$$x = c_1(t) \sin(\omega_D t) + c_2(t) \cos(\omega_D t).$$  \hspace{1cm} (11)

In Hayashi's treatment of almost-periodic oscillations, the behavior of $c_1$ and $c_2$ can be identified with a stable limit cycle on the graph of one coefficient versus the other. He considers oscillations in this class that develop from the harmonic oscillation ($\omega_D = \omega_0$), the higher-harmonic oscillation ($\omega_D \approx n\omega_0$), and the subharmonic oscillation ($\omega_D \approx n\omega_0/n$, where $n$ is an integer).

Adler [22] describes the transient pull-in process as well as the production of a distorted beat note using a differential equation for the oscillator phase as a function of time. He derives relationships between the observed average beat frequency $\Delta f$ and the conventional beat frequency $|f_D - f_0| \equiv \Delta f_0$,

$$\Delta f = \Delta f_0 \left[1 - (\Delta f_0/c)^2/(\Delta f_0)^2\right]^{1/2},$$  \hspace{1cm} (12a)

and

$$\Delta f = \Delta f_0 \left[1 - (M/M_c)^2\right]^{1/2},$$  \hspace{1cm} (12b)

where the subscript $c$ denotes the critical values just before entrainment occurs. The values of $\Delta f$ and $\Delta f_0$ deviate from being equal as the tendency toward synchronization produces strong harmonic distortion of the beat note.

III. EXPERIMENT

The experimental system is a circuit comprised of the unijunction transistor (RCA Electronics SK-9122), a capacitance $C$, an inductance $L$, and a resistance $R$. These discrete components are connected in series, as shown in Fig. 1. The UJT [24] has elements labeled $E$ for emitter, $B1$ for base 1, and $B2$ for base 2. The rectifying junction is between the emitter and the silicon substrate. The base terminals are Ohmic contacts. Current between $B1$ and $B2$ produces a voltage gradient along the substrate that can cause the emitter to be reversed biased.

The bias voltage $V_B$ is variable from zero to 12 V and the supply voltage $V_S$ is variable from zero to 40 V. The system’s response is measured as a voltage across $R$ with a digital oscilloscope (Philips PM 3350A) and a computer-controlled wave-form digitizer (LeCroy 6810) in a CAMAC crate. The inductively coupled driving force is provided by a synthesized-wave-form function generator (Wavelet 23). The ratio of the driving amplitude $M$ measured in the UJT oscillator circuit to the output of the function generator, referred to as the driving voltage, is $1 \times 10^{-3}$, as determined from the observed amplitude modulation in the circuit's oscillatory response for small driving amplitudes.

Electric current in the system oscillates spontaneously for values of $V_B$ greater than a critical value which is dependent on $V_S$. The precise value for the bias-voltage threshold depends on whether the voltage is being increased or decreased to find the transition. Spontaneous oscillation is observed throughout the range of bias voltage extending from this lower threshold value to an upper threshold value. No hysteresis is observed at the upper threshold value. The dependence of the oscillation amplitude $\text{bias}$ voltage is shown in Fig. 2. The hysteresis at the lower threshold value of bias voltage is displayed in the inset graph in Fig. 2. In the experiments reported here, the value $V_B = 4.34$ V is chosen for cases in which the bias voltage is held constant. At this value of bias voltage, the oscillation amplitude is $22 \text{ mV}_\text{rms}$.

The variation of spontaneous frequency with bias voltage and supply voltage is shown in Fig. 3. The spontaneous frequency is $5.8 \text{ kHz}$ for $V_B = 4.34$ V in Fig. 3(a) and also for $V_S = 38$ V in Fig. 3(b). This oscillation is not monochromatic. A common feature of the wave form is a shoulderlike deviation from a sinusoid that occurs each period. This deviation is associated with the noncircularity in the phase-space trajectory of the system's limit cycle. The wave form's time series, the wave forms phase-space trajectory, and the wave form's fast Fourier transform (FFT) are exhibited in Fig. 4. For the purposes of this paper, Fig. 4 establishes the spontaneous case as a
The degree of noncircularity in the wave form's phase-space trajectory changes when the oscillation amplitude and frequency change. The effect of changing these parameters on the sinusoidal character of the time series and the noncircularity of the phase-space trajectory is demonstrated in Fig. 5. Here, the bias voltage is ramped smoothly through the range for spontaneous oscillation in order to sweep the oscillation frequency as shown in Fig. 5(a). In Fig. 5(b), the time series is seen to decrease in amplitude and get less sinusoidal from left to right. The observable range of limit-cycle shapes for the spontaneous oscillation is pictured in Fig. 5(c) in a way that resembles the actual family of nested individual limit cycles.

The system oscillates at a frequency other than its spontaneous frequency when a periodically varying driving force is present with sufficient amplitude to entrain the system's response to the driving frequency. Entrained limit cycles have shapes that depend on frequency and amplitude in a way similar to that shown in Fig. 5, but note that while the system is entrained these two pa-

FIG. 3. Oscillation frequency's dependence on (a) bias voltage (for $V_b = 38 \, \text{V}$) and (b) supply voltage (for $V_b = 4.34 \, \text{V}$).

FIG. 4. The case of the undriven UJT oscillator (for $V_b = 4.34 \, \text{V}$, $V_s = 38 \, \text{V}$): (a) time series (1 megasample/sec), (b) phase-space trajectory of (a), and (c) frequency spectrum (resolution $\delta f = 100 \, \text{Hz}$).

FIG. 5. (a) Frequency sweep, (b) time series (0.5 megasample/sec), and (c) phase-space trajectory of (b) for UJT oscillation while the bias voltage is ramped smoothly through the range for spontaneous oscillation. The frequency in (a) is determined at each zero crossing in (b) by the inverse of the time difference between the previous and subsequent zero-voltage crossings. The uncertainty associated with this calculation is smaller than the datum symbol.
rameters can also be affected by the driving force.

The range of driving amplitudes capable of entraining the system has a minimum which varies with frequency and has no observed maximum. Figure 6 is a graph of this critical value of driving amplitude as a function of driving frequency for three different cases of supply voltage. Notice that the threshold value of driving amplitude becomes negligible at a different point for each case of supply voltage. These points are the frequencies of spontaneous oscillation as determined from Fig. 3(b) (5.8, 6.55, and 6.95 kHz for $V_s=38$, 25, and 12 V, respectively). For each of these points, the critical driving amplitude increases along an approximately straight line. The data suggest that the experimentally observed threshold curve has the same qualitative dependence on driving frequency as the threshold curve predicted by Eq. (10). For the experimentally relevant cases, Eq. (10) depends weakly on the parameter $J$ and strongly on the ratio $M/V_D$. This ratio is measured and then adjusted within experimental uncertainty to best fit the data in Fig. 6. The value $M/V_D=0.53 \times 10^{-3}$ is used for applying the model to the experimental data.

It is found experimentally that the ratio $(V_D)_{crit}/a_0$ remains constant for a given value of $\delta$. This scaling is consistent with that of the critical value of parameter $K$ on the right-hand side of Eq. (10). This scaling may account for the slight discrepancy between the data and the theory for driving frequencies below 5.5 kHz, in the case of $V_s=38$ V, since in that frequency range the oscillation amplitude is lower than it is for frequencies above 5.5 kHz, as will be seen in Fig. 7(c). For reasons that will be described later, the case of $V_s=25$ V is identified with $J=0$. For $V_s=12$ V, $J$ is chosen to be $-0.0126/e$ and for $V_s=38$ V, $J$ is chosen to be $+0.0231/e$.

The best evidence that the system under investigation can be classified as a forced van der Pol oscillator is the agreement between the observed driving-frequency dependence of the oscillation amplitude and that predicted by Eq. (6). Figure 7 shows the observed behavior of oscillation amplitude $b$ as a function of frequency for several values of driving voltage. The value of the supply voltage is different for each of the three cases presented: 12 [Fig. 7(a)], 25 [Fig. 7(b)], and 38 V [Fig. 7(c)]. The lower boundary of the region corresponding to entrainment is depicted in Fig. 7 with diamond symbols. The

**FIG. 6.** Frequency dependence of the minimum driving amplitude for which entrainment is observed ($V_s=4.34$ V), for three cases of supply voltage, $V_s$. Square, diamond, and circle symbols denote the cases $V_s=12$, 25, and 38 V, respectively. The solid lines are obtained from Eq. (10) using values of $f_0$ and $J$ appropriate to each experimental case. The value of $M/V_D$ used is $0.53 \times 10^{-3}$.

**FIG. 7.** Experimentally measured response curves of the UJT oscillator ($V_s=4.34$ V) for three cases of supply voltage $V_s$. The observed critical value of oscillation amplitude is denoted with the diamond symbol. The entrained oscillation amplitude for five cases of driving voltage $V_D=2$, 4, 6, 8, and 10 V is marked with cross symbols ($V_D=2$ V for the innermost curve and $V_D=10$ V for the outermost curve). The solid lines are obtained from Eq. (9) using values of $f_0$ [(a) 6.95, (b) 6.55, (c) 5.835 kHz] and $J$ [(a) $-0.22$, (b) 0, (c) $+0.40$] appropriate to each experimental case of supply voltage [(a) 12, (b) 25, (c) 38 V].
experimentally determined response curves are represented by crosses. The response curves on each graph terminate at the entrainment boundary, above and below the spontaneous frequency. Between each pair of end points a response curve reaches a maximum in oscillation amplitude. Predicted response curves from Eq. (6) corresponding to the lowest, middle, and highest driving-voltage-value cases in Fig. 7 are presented in Fig. 8. For reasons that will be described later, the value of $\epsilon$ is set to 0.06. The response curves in Figs. 7 and 8 corresponding to the same supply voltage have very similar patterns. Perhaps more striking is the similarity between theory and experiment in the way the supply-voltage parameter $J$ affects the pattern.

The patterns in both the experimental and theoretical sets of response curves are seen to change with supply voltage. For smaller values of supply voltage, the pattern leans slightly toward lower frequency from its spontaneous frequency. For larger values of supply voltage, the pattern leans slightly toward higher frequency. For $V_S = 25$ V, the pattern appears approximately symmetric about the spontaneous frequency. The data suggest that the value of the supply voltage influences the symmetry of this system's response curves in much the same way as the coefficient $\eta$ in Eq. (4) is found to influence the symmetry of the predicted response curves. For this reason, $\eta = 0$ (and, therefore, $J = 0$) is identified with $V_S = 25$ V, the case in which the pattern is approximately symmetric.

The predicted relationship between the amplitude maxima of the response curves and the driving frequency from Eq. (9) is represented in Fig. 7 as a solid line rising from the horizontal axis. The values of $J$ associated with each case of supply voltage are obtained by fitting the line from Eq. (9) to the amplitude maximum of the outermost ($V_D = 10$ V) response curve. The points at which this line crosses each of the other response curves correspond very well with the maxima of the curves. Furthermore, the observed driving-amplitude dependence of these response-curve maxima is consistent with the predicted dependence from Eq. (8), as shown in Fig. 9.

For driving amplitudes less than the entrainment-threshold value, the driving force produces a modulation in the system's oscillation. For the smaller driving amplitudes in this subthreshold range, the modulation resembles the conventional amplitude modulation typically experienced by a larger-amplitude sinusoid due to a smaller-amplitude sinusoid. For the larger driving amplitudes in this subthreshold range, the modulation is distinctly different and is associated with the phenomenon known as periodic pulling. The transition between one type of modulation and the other is gradual.

Figures 10, 11, and 12 present three sequences of cases, one in which the driving frequency, and two in which the driving amplitude are independently incremented. Each case contains the wave form's time series and FFT.

**FIG. 8.** Theoretically predicted response curves of a forced van der Pol oscillator appropriate to the experimental cases of Fig. 7. The value of $\epsilon$ used is 0.0576.

**FIG. 9.** Dependence of the entrained oscillation amplitude maximum on driving voltage ($V_S = 4.34$ V) for three cases of supply voltage. Square, diamond, circle symbols denote the cases $V_S = 12, 25$, and $38$ V, respectively. The solid line is obtained from Eq. (8). The value of $M/(\epsilon V_D)$ used is $9.2 \times 10^{-3}$. 
The most distinguishing feature of the periodic-pulling cases [cf., Figs. 10(d)–10(f), 11(d)–11(f), 12(d)–12(f), and 12(i)–12(k)] is the appearance of spectral features at frequencies incommensurate with the spontaneous or the driving frequencies. The spectral profile which includes these incommensurate frequencies is asymmetric, or single sided, with the extra spectral features present only on the side of the spontaneous frequency opposite to the driving frequency. In Figs. 13 and 14 the locations of the spectral features in Figs. 10, 11, and 12 are followed from the case of conventional-like amplitude modulation [cf., Figs. 10(a)–10(c), 11(a)–11(c), 12(a)–12(c), and 12(i)–12(k)], through the periodic pulling range, to entrainment [cf., Figs. 10(g) and 10(h), 11(g) and 11(h), and 12(g) and 12(h)]. The evolution of the spectrum is seen to take place on only one side of the driving frequency $f_D$ until synchronization occurs. The dotted, dashed, and solid lines in Figs. 13 and 14 indicate the evolution of the spontaneous frequency and the two sidebands that is predicted from a model of the forced van de Pol oscillator.

**FIG. 10.** Time series and frequency spectra of UJT oscillations ($V_s=4.34$ V, $V_f=38$ V, $f_D=5.250$ kHz) for a sequence of cases with different driving voltages. $V_D$ (in volts) = (a) 0.55, (b) 1.10, (c) 1.25, (d) 1.70, (e) 1.85, (f) 2.10, (g) 2.30, (h) 4.60.

**FIG. 11.** Time series and frequency spectra of UJT oscillations ($V_s=4.34$ V, $V_f=38$ V, $f_D=6.275$ kHz) for a sequence of cases with different driving voltages. $V_D$ (in volts) = (a) 0.40, (b) 0.80, (c) 1.10 (d) 1.40, (e) 1.70, (f) 2.00, (g) 2.30, (h) 4.60.
that does not include periodic pulling. Notice that the observed evolution of the spontaneous frequency and the sideband opposite \( f_D \) does not show the model's predicted sharp jump from the conventional-like AM case to the entrained case.

Another distinguishing feature of the periodic pulling cases is the nonsinusoidal modulation of the wave-form's oscillation envelope. This feature is reflected in the phase-space trajectory by the brief excursions of the trajectory away from the primary limit cycle. These deviations from the behavior of the entrained case result from the cyclic interruption of the system's coupling to the driving force. Another example of nonsinusoidal modulation of a wave form due to periodic pulling is seen in Fig. 15. For driving-force parameters slightly different from the cases shown in Figs. 10, 11, and 12, the modulation in Fig. 15 is occurring in a more obviously pulsed fashion. Here, periodic pulling is causing nearly 100% amplitude modulation of the wave-form envelope, even though the driving amplitude is very small compared to the spontaneous oscillation amplitude. In Fig. 16(a), the frequency of oscillation and phase relationship to the driving force in Fig. 15(a) are seen to evolve between these points of 100% amplitude modulation. Initially the system appears to oscillate at or near the spontaneous frequency (5.835 kHz). In less than four cycles, the frequency is pulled up from 14% below \( f_D \) to approximately 1.5% below the driving frequency. During the evolution of the oscillation frequency, a phase mismatch approaching 180° develops between the driving force and the system's response, as illustrated in Fig. 16(b). Adler's [22] statement that the phase lag is zero when the system

**FIG. 12.** Time series and frequency spectra of UJT oscillations \( (V_S = 4.34 \text{ V}, V_S' = 38 \text{ V}, V_D = 2.0 \text{ V}) \) for a sequence of cases with different driving frequencies. \( f_D \) (in kHz) = (a) 4.750, (b) 4.850, (c) 4.950, (d) 5.050, (e) 5.150, (f) 5.250, (g) 5.350, (h) 6.250, (i) 6.350, (j) 6.450, (k) 6.550, (l) 6.650, (m) 6.750, (n) 6.850.
oscillates at $f_0$ is used to set the zero position on the vertical scale of Fig. 16(a) and to horizontally position the sinusoid in Fig. 16(b). Notice that the phase shifts faster when the difference between the oscillation and driver frequencies is larger. The incomplete synchronization is suddenly terminated, and this process, which involves both AM and FM, is repeated. Analyzing the wave forms in Figs. 10, 11, and 12 to obtain the time-dependent frequency and amplitude of the system reveals behavior [33] very similar to that shown in Fig. 16.

The frequency of the beats in the wave-form envelope caused by periodic pulling has a value [500 Hz for Fig. 15(a)] that is different from the conventional beat frequency $|f_D - f_0|$ [≈900 Hz for Fig. 15(a)]. More distinctive is the sensitivity of the overall dependence of the periodic pulling case's beat frequency on the amplitude and frequency of the driving force, compared to that of the conventional beat frequency.

For conventional amplitude modulation, the beat frequency is independent of the two amplitudes involved and linearly proportional to the difference in the two frequencies involved. In contrast, Fig. 17 shows the observed beat frequency and the predicted beat frequency for periodic pulling cases in which the driving frequency is fixed and the driving amplitude is varied. Figure 18

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**FIG. 13.** Evolution of the spectral content of the UJT oscillations shown in Figs. 10 and 11. Each point indicates the presence of a distinct spectral feature in the fast Fourier transform. For $V_D < 1$ V, features are present at $f_D$, $f_0$ and $2f_0 - f_D$ only. For $V_D > 2.1$ V, a single feature is present at $f_D$. $f_B = (a)\ 5.250$ and (b) 6.275 kHz. The lines are explained in the text.

**FIG. 14.** Evolution of the spectral content of the UJT oscillations shown in Fig. 12. Each point indicates the presence of a distinct spectral feature in the fast Fourier transform. Inside the range $5.3 < f_D < 6.3$ kHz, a single feature is present at $f_D$. For the regions $4.5 > f_D > 7$ kHz, features are present at $f_D, f_0$, and $2f_0 - f_D$ only. The lines are explained in the text.

**FIG. 15.** Example of periodic pulling with 100% amplitude modulation ($V_D = 4.34$ V, $V_c = 38$ V, $f_D = 6.750$ kHz, $V_D = 1.88$ V): (a) time series (0.1 megasample/sec), (b) phase-space trajectory of (a), (c) frequency spectrum of (a) (resolution $\delta f = 100$ Hz).
shows the observed beat frequency and the predicted beat frequency for periodic pulling cases in which the driving amplitude is fixed and the driving frequency is varied. The data in Figs. 17 and 18 come from the sequences of cases shown in Figs. 10, 11, and 12. The experimentally determined curves agree well with the curves predicted by Eq. (12) (solid lines) for the forced van der Pol oscillator model that includes periodic pulling [22]. In Figs. 17 and 18, dashed lines represent the predicted dependence for the standard forced van der Pol oscillator that includes entrainment but no periodic pulling. Here, periodic pulling is seen to significantly change the beat frequency of the system compared to conventional expectations over 75% of the driving-amplitude range below entrainment and over at least 25% of the driving-frequency range below entrainment.

IV. DISCUSSION

The UJT oscillator circuit investigated here can be classified as a forced van der Pol oscillator because its response, when entrained to a low-amplitude driving force, is shown to closely resemble that described by the forced van der Pol equation. The effect of an adjustable UJT supply voltage on this response is shown to be similar to the effect of an adjustable nonlinear restoring force on the response of a forced van der Pol oscillator.

Periodic pulling is shown to occur in this system for

![Figure 17](image17.jpg)

**FIG. 17.** Observed and predicted beat frequency as a function of driving voltage ($V_b=4.34$ V, $V_S=38$ V) for the cases shown in Figs. 10 and 11: $f_b =$ (a) 5.250 and (b) 6.275 kHz. The solid line is obtained from Eq. (12b) using (a) $\Delta f_0=0.585$ kHz and $(V_D/M)M_0=2.2$ V, and (b) $\Delta f_0=0.440$ kHz and $(V_D/M)M_0=2.0$ V. The dashed line is obtained from the forced van der Pol model with entrainment but no periodic pulling.

![Figure 16](image16.jpg)

**FIG. 16.** (a) Evolution of the UJT oscillation frequency (solid circles) and the phase mismatch between the UJT oscillation and the driving force (open circles, dotted line) during the second beat note in Fig. 15(a). (b) Illustration of the phase mismatch developing during the second beat note in Fig. 15(a). Notice how the relative phase between the system's response (solid line) and the driving force (dotted line) changes suddenly from out of phase to in phase between beat notes. Points in (a) are located at each zero crossing of the time series in Fig. 15(a) and represent averages over the oscillation period on which each is centered (cf. Fig. 5).

![Figure 18](image18.jpg)

**FIG. 18.** Observed and predicted beat frequency as a function of driving frequency ($V_b=4.34$ V, $V_S=38$ V, $V_D=2$ V) for the cases shown in Fig. 12. The solid line is obtained from Eq. (12a) using $(\Delta f_0)= -0.530$ and 0.436 kHz. The dashed line is obtained from the forced van der Pol model with entrainment but no periodic pulling.
values of driving amplitude and frequency just below those required to entrain the UJT oscillator. Periodic pulling results in nonsinusoidal amplitude and frequency modulation of the oscillatory wave form, periodic evolution of the limit cycle in phase space, and the introduction of frequencies incommensurable with \( f_0 \) and \( f_D \) in the spectrum, all of which are observed. The transition from conventional-like amplitude modulation, through periodic pulling behavior, to entrainment as the driving force is varied is presented in detail. The dependences of observed beat frequency on driving frequency and amplitude agree with the prediction of Adler [22].

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