Time dependent evolution of linear kinetic Alfvén waves in inhomogeneous plasma

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The propagation of linear Kinetic Alfvén waves (KAWs) in inhomogeneous magnetized plasma has been studied while including inhomogeneities in transverse and parallel directions relative to the background magnetic field. The propagation of KAWs in inhomogeneous magnetized plasma is expected to play a key role in energy transfer and turbulence generation in space and laboratory plasmas. The inhomogeneity scale lengths in both directions may control the nature of fluctuations and localization of the waves. We present a theoretical study of the localization of KAWs, variations in magnetic field amplitude in time, and variation in the frequency spectra arising from inhomogeneities. The relevance of the model to space and laboratory observations is discussed.

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I. INTRODUCTION

Alfvén waves are low frequency waves often observed in space plasmas. They are the means by which magnetized plasmas convey internal information about magnetic fields and changing currents.1 These waves may play an important role in transport of energy, in driving field aligned currents, and in particle acceleration and heating.2 Understanding and characterizing the properties of Alfvén waves has been of interest to many researchers.3-6 Because Alfvén waves have such long wavelengths, it is very difficult to excite them in laboratory experiments. Wilcox et al.7 successfully measured waves propagating at the Alfvén speed ($V_p^2 = B_0^2/4\pi n_0 m_i$) in a laboratory plasma experiment; where $B_0$ is the background magnetic field, $n_0$ is the background plasma density, and $m_i$ is the ion mass. In space and laboratory plasmas, finite-amplitude Alfvén waves are excited by many sources such as energetic charged particle beams, non-uniform background plasma parameters,3 and electromagnetic waves.8

If Alfvén waves exhibit a large wave number ($k_L$) in transverse direction to the background magnetic field $B_0$, they are categorized as kinetic Alfvén waves (KAWs). KAWs are dispersive and experience a variety of nonlinear effects that are not observed in simple Alfvén wave propagation.3-6,9 KAWs are believed to play important roles in many space phenomena, such as heating of solar coronal loops, solar flares, and the dissipation of solar wind turbulence.10 Recent observations11 suggest that low frequency electromagnetic fluctuations in the auroral ionosphere and magnetosphere may also result from KAWs. These observations have motivated studies designed to understand how KAW dissipation might heat and accelerate plasma particles12 and how KAWs may be excited by magnetohydrodynamic (MHD) waves.13 Departure from ideal MHD enables KAWs to exist as a normal mode of the plasma. KAW is not a purely electromagnetic wave and does not propagate strictly along the magnetic field. Hollweg14 examined the properties of KAW with emphasis on the nonlinear properties of KAWs. Such nonlinear effects have significant effects in theoretical models2,15,16 as well as experiments.17 Renewed emphasis on the likely role of KAWs in space plasmas has led to many laboratory investigations of KAWs in the Large Plasma Device LAPD at UCLA18 and recent experiments in the Hot hElicon eXperiment (HELIX)19 at WVU. The LAPD experiments include studies of both high power and linear regimes of KAW propagation.

A variety of inhomogeneities, e.g., temperature, magnetic field, and density, are present in both space and laboratory plasmas. Gekelman et al.20 investigated the cross-field propagation of shear Alfvén waves excited by small scale (inhomogeneous) currents driven along the background magnetic field direction. Amagishi et al.21 excited Alfvén waves through imposed density inhomogeneities in cylindrical plasma. Rauf and Tataronis22 employed an inhomogeneous magnetic field to create a longitudinal current through the shearing of small amplitude Alfvén waves. Houshmandyar and Scime23 studied the propagation of KAW and inertial Alfvén waves in magnetized plasma with steep radial density gradients. Building upon the Houshmandyar and Scime experimental results, Sharma et al.24 performed a numerical study of the generation of kinetic Alfvén turbulence arising from density inhomogeneities perpendicular to the background magnetic field. Besides this, the inhomogeneous plasmas can be source for electrostatic potential (i.e., spatially localized dc electric fields) that are likely to accompany steep density gradients and contribute to the inhomogeneity. The localized transverse electric fields can significantly affect wave dispersion properties and can be a source for Alfvénic waves25-27 which may amplify the antenna-launched waves. Scime et al.28 have observed high speed flows that are believed to be due to this potential structure.

Missing from the kinetic Alfvén turbulence study was any time-dependent physics, i.e., the model was restricted to steady state systems and turbulence features in time domain.

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could not be explained. Actually, Sharma et al.\textsuperscript{24} considered steady state model, therefore the spatio-temporal features of the magnetic field evolution could not be explained. These kind of features have indeed been observed in WVU experiment.\textsuperscript{23} In addition, the restriction of the model to initial wave field profiles that were purely Gaussian, limited the predictions for the turbulence spectra arising from different inhomogeneities to the wave number domain. This model was able to describe steady state turbulence spectra for different inhomogeneities but as the experiment in WVU describes magnetic field fluctuations in time domain, this necessitates having the insight of physics behind the experiment used to study spatio-temporal phenomena. So, the understanding of the magnetic field fluctuations in the time domain requires development of a more detailed model. This is the main objective of the present work. Therefore, we have first developed the model equations applicable to transient case and then solved them numerically. Although this model is also a very preliminary model and will need further work for improvement in order to do the exact comparison with the WVU transient observations of magnetic field fluctuations in space and time.

The first Bessel function-like fluctuation amplitude radial profile reported by Houshmandyar and Scime\textsuperscript{23} is reasonably well approximated with a hollow Gaussian function as shown in Fig. 1. The model described in this work assumes a hollow Gaussian radial profile for the initial wave magnetic field amplitude and explores the transient behavior of propagating, linear, KAWs in an inhomogeneous plasma by taking into account density inhomogeneity in the transverse and parallel directions relative to the background magnetic field. The magnetic field fluctuations in the time domain and the power spectra of the waves in the frequency domain are calculated for the same inhomogeneity scale lengths used previously.\textsuperscript{24} The analysis begins with a discussion of the numerical simulation technique employed to study the localization of linear KAWs and then a semi-analytical model is developed to study the propagation of KAWs in inhomogeneous plasma under the paraxial approximation.

In Sec. II, the dynamical equations for linear propagating KAWs are described and then solved numerically. In Sec. III, the results of the simulation of the dynamics of KAWs are presented. Conclusions and additional result are summarized in Sec. IV.

II. KAW DYNAMICS

Consider an inhomogeneous magnetized plasma having magnetic field $\vec{B}_0$ in z direction and density inhomogeneity in both x and z directions having the profile $n_0 \exp\left(-x^2/L_x^2 - z^2/L_z^2\right)$, where $L_x$, $L_z$ is the inhomogeneity scale length in transverse (parallel) direction to $\vec{B}_0$. The initial profile of the KAW amplitude is assumed to follow a hollow Gaussian profile, which approximates the first Bessel function profile up to the first root.\textsuperscript{24} Fig. 1 shows both functions on the same scale.

Following the standard method\textsuperscript{29–31} and using Maxwell’s equations, the continuity equation and the equation of motion, the dynamical equation for finite amplitude, low frequency KAWs propagating in x-z plane is

$$\frac{\partial^2 B_z}{\partial t^2} = -(V_x^2 + V_z^2) \frac{\partial^2 B_z}{\partial x^2} + V_z^2 \exp\left(x^2/L_x^2 + z^2/L_z^2\right) \frac{\partial^2 B_z}{\partial z^2}, \quad (1)$$

where $V_T = \sqrt{T_e/m_e}$ is the electron thermal speed, $V_i = \sqrt{T_i/m_i}$ is the ion thermal speed. $T_e$ ($T_i$) is the temperature of the electrons (ions). $\lambda_e = \sqrt{e^2 m_e/4\pi\rho}\omega_e^2$ is the collisionless inertial length of the electrons, $\rho_i = V_i/\omega_i$ is the ion gyroradius, and $\omega_i$ is the ion cyclotron frequency.

To study the localization process, we substitute the envelope solution

$$B_z = \tilde{B}_0(x, z, t) e^{(i\omega_0 + k_0x + k_2z - \epsilon)} \quad (2)$$

into Eq. (1) and obtain

$$\frac{2i\omega_0}{V_x^2 A^2_{0z} x^2} \frac{\partial^2 \tilde{B}_0}{\partial t^2} + \frac{2i}{k_0} \frac{\partial \tilde{B}_0}{\partial z} + \left(\frac{V_x^2}{V_A^2} \frac{k_0^2}{x^2} \frac{\partial^2 \tilde{B}_0}{\partial x^2} - \tilde{B}_0 \exp\left(x^2/L_x^2 + z^2/L_z^2\right) + \tilde{B}_0 = 0, \quad (3)$$

where $\partial \tilde{B}_0 \ll k_0 \tilde{B}_0$ and $\partial \tilde{B}_0 \gg k_0 \tilde{B}_0$. When normalized and written in dimensionless form, Eq. (3) becomes

$$\frac{\partial B}{\partial t} + \frac{\partial^2 B}{\partial x^2} + \frac{\partial B}{\partial z} + \left(1 - \exp\left(\frac{x^2}{\xi_x} + \frac{z^2}{\xi_z}\right)\right) B = 0, \quad (4)$$

where $\varphi_x = x^2/\lambda_x^2$ and $\varphi_z = z^2/\lambda_z^2$ are constants and the normalizing parameters: $x_n = \sqrt{(V_x^2 + V_z^2)/V_A^2}$, $z_n = 2/k_0$, $t_n = 2\omega_0/\sqrt{V_A^2}$, $\xi_x = L_x/\lambda_x$ and $\xi_z = L_z/\lambda_z$ are the normalized inhomogeneity scale lengths of the plasma channel. $\lambda_x (\lambda_z)$ is the wavelength of KAW in transverse (parallel) direction to the background magnetic field. $k_0 (k_0)$ is the component of the wave vector perpendicular (parallel) to the background magnetic field and $\omega_0$ is the frequency of the KAW. Based on the motivating experiment conducted in helium plasma, in our semi-analytic analysis and our numerical studies, we have assumed the following typical

![FIG. 1. The initial wave field following Hollow Gaussian profile and truncated first Bessel’s function up to first root showing nearly the same behavior where x is in dimensionless units.](image)

laboratory plasma parameters: \( B_0 \approx 560 \) G, \( n_0 \approx 10^{13} \) cm\(^{-3} \), plasma density \( \approx 6 \times 10^{12} \) cm\(^{-3} \), \( T_e = 81200 \) K, \( T_i = 2900 \) K. These values yield characteristic plasma parameters of: 
\[
\beta \approx 0.002, \quad V_A \approx 2.5 \times 10^7 \text{ cm/s}, \quad V_{Te} \approx 1.1 \times 10^7 \text{ cm/s}, \quad V_{T_i} \approx 2.445 \times 10^6 \text{ cm/s}, \quad c_s = 1.3 \times 10^6 \quad \text{ cm/s}, \quad \omega_{ci} = 2.68 \times 10^6 \text{ rad/s}, \quad \lambda_e = 0.1085 \text{ cm}, \quad \rho_i \approx 0.0912 \text{ cm}, \quad \rho_e \approx 0.483 \text{ cm}, \quad k_0 \approx 0.85 \text{ cm}^{-1}, \quad k_{0z} \approx 0.025 \text{ cm}^{-1}, \quad \omega_{0z}/\omega_{ci} = f_0/f_{ci} = 0.4, \quad \text{and} \quad \omega_0 = 1.07 \times 10^6 \text{ rad/s}. \]
Our calculations explore three different inhomogeneity scale lengths in transverse direction \((L_x)\) for a single, fixed, inhomogeneity scale length in longitudinal direction \((L_z)\) along the background magnetic field. For the parameters defined above, \( x_n \approx 0.486 \text{ cm} \) and \( \lambda_e \approx 7.40 \text{ cm} \) for \( k_0z \approx 0.85 \text{ cm}^{-1}, \quad z_n \approx 80 \text{ cm} \) and \( \lambda_z \approx 251.32 \text{ cm} \) for \( k_{0z} \approx 0.025 \text{ cm}^{-1}, \quad t_n \approx 5.48 \times 10^{-5} \text{ s} \) for \( \omega_{0z}/\omega_{ci} = f_0/f_{ci} = 0.4, \quad \text{and} \quad \omega_0 = 1.07 \times 10^6 \text{ rad/s}. \)
We want to solve Eq. (4) numerically, so we adopt the algorithm developed for solving the Non Linear Schrödinger (NLS) equation. Therefore, after testing the invariants of NLS up to \( 10^{-3} \) accuracy, we modified the algorithm for solving the NLS to accommodate Eq. (4). So, Eq. (4) is mathematically similar to the NLS equation if we neglect the third term and replace the last term on left hand side by \((B^* B) B\). For the initial hollow Gaussian profile

\[
B_0(x, z, t) = a_0 \left( x^2/r_{10}^2 + z^2/r_{20}^2 \right) \exp(-x^2/r_{10}^2 - z^2/r_{20}^2),
\]
where \( a_0 \) is the initial value of the amplitude of the KAW and \( r_{10}(r_{20}) \) is the scale size of the initial wave field in the transverse (parallel) direction to the background magnetic field, Eq. (4) is solved numerically. To control the dynamics of KAW evolution, we choose \( a_0 = 0.02 \) and \( r_{10} = r_{20} = 5 \) as parameters where \( a_0 \) has been chosen using the experimental setup, which defines the wave launching field normalized by background magnetic field \( B_0 \). \( r_{10} \) and \( r_{20} \), respectively, define the system length to cover the size of Hollow Gaussian profile in transverse and parallel direction to \( B_0 \). Equation (4) was solved using the method adopted by Sharma and Singh\(^{22}\) based on a pseudo spectral method for space integration with periodic lengths \( l_x = r_{10}, \quad l_z = r_{20} \) and \( 128 \times 128 \) grid points. A finite difference method with predictor corrector scheme was employed for the evolution in time. The time step was taken to be of the order of \( \Delta t \approx 10^{-3} \) s.

**III. SIMULATION RESULTS AND SIMPLIFIED MODEL**

The normalized magnetic field intensities as a function of time and inhomogeneity scale length are shown in Figs. 2(a)-2(c). As the transverse inhomogeneity scale length increases, the radial profile broadens and at later times it splits into peaks of different intensities. The magnetic field intensity (normalized) initially increases and then decreases as time elapses, but it is always greater than the initial peak at \( t = 0 \); which demonstrates the effect of initial wave field profile on the linear KAW.

The temporal evolution of the magnetic field fluctuations is shown in Fig. 3 on a time window of 300 \( \mu \)s. It should be mentioned here that the actual time duration was around 600 \( \mu \)s but in order to have a proper view, we have shown the truncated case in Fig. 3. The variations in the normalized magnetic field intensity variation of KAW with direction of propagation (z) and transverse direction (x) at different times for inhomogeneity scale length \( \zeta_z = 0.6 \), where \( z, x, \) and \( t \) are normalized as defined for Eq. (3) to obtain Eq. (4). The normalized magnetic field intensity variation of KAW with direction of propagation (z) and transverse direction (x) at different times for inhomogeneity scale length \( \zeta_x = 0.9 \), where \( z, x, \) and \( t \) are normalized as defined for Eq. (3) to obtain Eq. (4).
magnetic field are shown for a fixed location (x, z) of the mid-point on the longitudinal axis (z) and nearly the end point on the transverse axis (x); point (31.4, 15.7) in normalized units. Although the magnetic field fluctuates in time for all the three inhomogeneity scale lengths, there are significant differences in the amplitude and frequency of the fluctuations (Fig. 3).

Fig. 4 shows the variation of $|B_0|^2$ with $\omega$, the truncated power spectra, which has been obtained using the Fourier transform of the total magnetic field fluctuations (for 600 $\mu$s) and has been shown in smaller window to have a clear view. The different spectra show the levels of turbulence versus frequency for the different values of inhomogeneity scale length. At higher frequencies, the slope of the spectra increases slightly with decreasing inhomogeneity scale length. The effect of inhomogeneity is to change the level of turbulence. The spectrum with smallest inhomogeneity scale length also has a pronounced peak at a specific low frequency. The peak suggests a maximum localization effect at smallest inhomogeneity scale length considered.

Because of the finite inhomogeneity scale length $L_z$, the localization changes with time as described by Eq. (4) and shown in Figs. 2(a)–2(c). The spectral components of the spatially localized structures are expected to evolve at different rates throughout time. Therefore, the magnetic field fluctuation amplitude is expected to vary with time at a particular spatial location as shown in Fig. 3 and the Fourier transform of the evolving magnetic field yields the frequency spectra shown in Fig. 4.

To understand the physics responsible for the numerical simulation results, we have developed a simplified model by using semi-analytical approach and the paraxial approximation; the plasma is assumed to be homogeneous along the z direction and the only inhomogeneity is transverse to background magnetic field ($L_z \gg L_x$). We approximate Eq. (1) as

$$\frac{\partial^2 B_z}{\partial t^2} = -(V_T^2 \tau^2 + V_A^2 \rho_i^2) \frac{\partial^2 B_z}{\partial x^2} + V_A^2 (1 + x^2/L_x^2) \frac{\partial^2 B_z}{\partial x^2}.$$  

(6)

Equation (6) is solved assuming the paraxial approximation ($x \ll r_0 f_0$), $r_0$ is the scale size of the KAW in the transverse direction and $f_0$ is a dimensionless parameter that quantifies the beam width of the KAW. Substituting the envelope solution Eq. (2) into Eq. (6), the steady state KAW equation becomes

$$-2i k_0 f_0 \frac{\partial B_0}{\partial z} - k_0^2 V_A^2 (V_T^2 \tau^2 + V_A^2 \rho_i^2) \frac{\partial^2 B_0}{\partial x^2} + k_0^2 \left( \frac{x^2}{L_x^2} \right) B_0 = 0.$$  

(7)

An additional eikonal $S_0$ is introduced to separate Eq. (7) into real and imaginary parts by assuming a solution of the form

$$B_0 = \tilde{A}_0(x, z) \exp \{i k_0 S_0(x, z) \}.$$  

(8)

Solution of Eq. (7) is accomplished with a technique analogous to the paraxial approximation in which Eqs. (7) and (8) are expressed in terms of parameters $\eta$ and $z$, where

$$\eta = \frac{x}{r_0 f_0} - \sqrt{2},$$  

(9)

$r_0 f_0$ is the beam width, and $x = \sqrt{2} r_0 f_0$ is the position of maximum irradiance.

The solution of the real part is then

$$\tilde{A}_0^2(\eta, z) = \tilde{B}_0^2 f_0 \left( \frac{\sqrt{2} + \eta}{2} \right) \exp \left( -\sqrt{2} + \eta \right) / 2$$

$$S_0(\eta, z) = \left( \frac{\sqrt{2} + \eta}{2} \right) z + \phi_0$$

$$z(z) = a r_0^2 f_0 \frac{df_0}{dz},$$

(10)

where $a = \{(\rho_i^2 + \rho_e^2) k_0^2 \}^{-1}$, $\rho_i = c_i / a c_i$ is the ion gyroradius at the electron temperature, $c_i = \sqrt{T_e / m_i}$ is the ion sound speed, and the governing differential equation for the KAW beam width parameter $f_0$ is

$$\frac{d^2 f_0}{dz^2} = \frac{7}{12 a^2 R_i^2} - \frac{f_0 R_i^2}{a L_x^2}.$$  

(11)

where $R_i = k_0 r_0^2$.

The right hand side of Eq. (11) contains the terms arising from diffraction and the inhomogeneous plasma profile. If these terms balance each other, self-trapping of the KAW.
occurs. In that case, the convergence or divergence of the wave does not occur (i.e., $f_0 = 1$) as it propagates along $z$. This gives us a critical transverse inhomogeneity scale length ($L_x^2(Cr_0) \sim (12/7) a R_d^2$), where $R_d = 10 \text{ cm}$ for $r_0 \sim 20 \text{ cm}$.

To solve Eq. (11) numerically, we assume boundary conditions corresponding to a plain wave, i.e., $f_0 = 1$ and $df_0/dz = 0$ at $z = 0$. According to Eq. (11), if the inhomogeneity scale length of the plasma channel is greater than a critical value ($L_x^2(Cr_0) \sim (12/7) a R_d^2$), the last term on the right hand side dominates over the first term. In that case, the value of the beam width $f_0$ reduces continuously with the propagation until the first term, i.e., diffraction, starts to dominate. Thus, an intensity maximum arises at the point where $f_0$ has a minimum value. Beyond the intensity maximum, due to domination of the diffraction and the inhomogeneity terms alternatively, there is an increase and decrease in the value of $f_0$. The variation of beam width parameter $f_0$ along the direction of propagation is shown in Fig. 5. Thus, the filaments identified in the numerical simulation arise because the competition between the diffraction and inhomogeneous terms results into focusing and defocusing of the KAW beam and give rise to the filaments that appear in Figs. 2(a)–2(c).

Figures 2(a)–2(c) clearly indicate that as the inhomogeneity scale length increases, the magnetic field intensity not only decreases but may split into multiple peaks also. The decrease in intensity is also predicted by the simplified (paraxial) model given by Eq. (11). If the value of inhomogeneity scale length $L_x$ increases, the second term in Eq. (11) decreases. The result is to force $f_0$ to diverge to larger values and hence the normalized magnetic field intensity decreases. Thus, we observe a change in the radial structures for a particular fixed time due to variations in the value of inhomogeneity scale length $L_x$ consistent with Figs. 2(a)–2(c). It should be mentioned here that the temporal behavior shown in Figs. 2(a)–2(c) is beyond the scope of simplified model discussed above. A model needs to be developed to explain the temporal features of magnetic field intensity. However, the localization based on inhomogeneity scale length at a given time can be understood using Eq. (11).

![Graph](image)

**FIG. 5.** The variation of beam width parameter ($f_0$) of KAW along the direction of propagation ($z$) where $z$ is normalized by $R_d$.

**IV. CONCLUSION**

We have studied the localization of KAWs in inhomogeneous plasma when the inhomogeneity is transverse as well as parallel to the background magnetic field. This model is modified from our previous model representing the turbulence in laboratory plasma in steady state only. The present model provides insight in the spatio-temporal behavior of the wave magnetic field and provides an overview of origin of fluctuation in frequency domain when inhomogeneity scale length is changed. These features could not be explained by our previous work as reported in Sharma et al. Even the present work makes use of fluid model of the plasma. Kinetic effects like Landau damping have not been incorporated in the model. Landau damping may affect the localization process and the turbulence level. These are some of the limitations. We shall consider these in our future work.

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