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First order perturbed velocity distribution theory and measurement

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Definition of Terms

Ω = cyclotron frequency for both ions and electrons
M = mass for both ions and electrons
ω = wave frequency
$k_\perp$ = perpendicular wave number
$k_{||}$ = parallel wave number
k_{ds} = Debye wave number for species s
Φ = Electrostatic potential (wave amplitude)
$v_{t\perp}$ = perpendicular thermal velocity
$v_{t||}$ = parallel thermal velocity

$_\perp$ = perpendicular with respect to ambient background magnetic field
$_{||}$ = parallel with respect to ambient background magnetic field

1 Introduction

In 1987, F Skiff and F. Anderegg[1] published the first paper theorizing the possibility of measuring wave numbers from a propagating wave using Laser Induced Florescence (LIF).[2,3] Due to the non-local nature of the plasma dielectric function, a local measurement provides information about the response of the plasma due to electrostatic or electromagnetic waves. Using the velocity and spatial resolution of LIF measurements, the perturbed ion velocity distribution due to a wave can be measured. From linear Vlasov theory the method of characteristics[4,5,6] can be used to calculate the perturbed velocity distribution function for ions (and electrons). Using the theory to fit the experimental data reveals the wave number information for a wave. The advantage to this method, as with all LIF measurements, is clearly the fact that the measurement is non-intrusive, thus measuring the wavelengths of waves without affecting the plasma.

This document develops the theoretical, as well as, the experimental technique for measuring wave numbers using LIF. The theory for electrostatic waves will be outlined in section II, a generalization to electromagnetic waves will be developed in section III, section IV will contain experimental details for measuring the perturbation to the distribution function, and section V will comment on other applications using the $f_1(\mathbf{r},v,t)$ measurement.

2 Electrostatic waves

Safarty[7] *et. al.* published the first paper with experimental results measuring the first order perturbation to the distribution function and calculating the wave numbers for electrostatic waves. This was done using an antenna to launch a wave in the plasma and using LIF to collect the information about the wave. Electrostatic waves were used because they are the easiest waves to launch and have the easiest theory as will be seen later in the discussion of electromagnetic waves. Starting from the collisionless Vlasov equation, a method of integrating over the unperturbed orbits known as the method of characteristics[4,5,6] is used to calculate the first order perturbation to the velocity distribution function. Since the first order perturbation to the velocity distribution function is used to calculate the hot plasma dielectric tensor, the derivation follows closely to the development of the hot plasma dielectric tensor calculations done by Swanson[8].

The linear Vlasov Equation with no collisions is

$$\frac{df(\vec{r},\vec{v},t)}{dt} = \frac{\partial f(\vec{r},\vec{v},t)}{\partial t} + \vec{v}\cdot\nabla f(\vec{r},\vec{v},t) + \frac{q}{M}[\vec{E}(\vec{r},t)\cdot\nabla_v f(\vec{r},\vec{v},t) + \vec{v}\times\vec{B}(\vec{r},t)\cdot\nabla_v f(\vec{r},\vec{v},t)] = 0 \tag{1}$$

This eqation is the conservation of particals in time where total time derivative for the six dimensional phase space is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla + \vec{a} \cdot \nabla_v \tag{2}$$

where v is the velocity and a is acceleration. The Vlasov equation can be linearized in the following way

$$f(\vec{r}, \vec{v}, t) = f_o(\vec{r}, \vec{v}) + f_1(\vec{r}, \vec{v}, t) \tag{3}$$

and equation (1) can then be rewritten as

$$\frac{df}{dt} = \frac{df_0}{dt} + \frac{df_1}{dt} = \left\{ \frac{\partial f_0}{\partial t} + \vec{v} \cdot \nabla_x f_0 + \frac{q}{M}[\vec{E}_o + \vec{v} \times \vec{B}_0] \cdot \nabla_v f_o \right\} + \left\{ \frac{\partial f_1}{\partial t} + \vec{v} \cdot \nabla_x f_1 + \frac{q}{M}[\vec{E}_o + \vec{v} \times \vec{B}_o] \cdot \nabla_v f_1 \right\} + \frac{q}{M}[\vec{E}_1 + \vec{v} \times \vec{B}_1] \cdot \nabla_v f_o + \frac{q}{M}[\vec{E}_1 + \vec{v} \times \vec{B}_1] \cdot \nabla_v f_1 = 0 \tag{4}$$

Here the first term in brakets is df_0/dt, the zeroth order vlasov equation, which is zero since the plasma is asummed to be in an equilibrium state. The second term in brakets is df_1/dt, the first order vlasov equation. Note that since v is an independent variable, or since there is no equilibrium velocity as in fluid theory, velocity, is not linearized. If we neglect the second order term (the last term), equation (4) becomes

$$\frac{df_1(\vec{r}, \vec{v}, t)}{dt} + \frac{q}{M}[\vec{E}_1(\vec{r}, t) + \vec{v} \times \vec{B}_1(\vec{r}, t)] \cdot \nabla_v f_o(\vec{r}, \vec{v}) = 0 \tag{5}$$

Solving for $f_1(\mathbf{r},\mathbf{v},t)$ equation (5) becomes

$$\frac{df_1(\vec{r}, \vec{v}, t)}{dt} = -\frac{q}{M}[\vec{E}_1(\vec{r}, t) + \vec{v} \times \vec{B}_1(\vec{r}, t)] \cdot \nabla_v f_o(\vec{r}, \vec{v}) \tag{6}$$

or

$$f_1(\vec{r}, \vec{v}, t) = -\frac{q}{M} \int_{t_o}^{t} dt' [\vec{E}_1(\vec{r}\,', t') + \vec{v} \times \vec{B}_1(\vec{r}\,', t')] \cdot \nabla_v f_o(\vec{r}\,', \vec{v}\,') \quad + f_1(\vec{r}, \vec{v}, t_o) \tag{7}$$

Equation (7) is the general form for $f_1(\mathbf{r},\mathbf{v},t)$ with the initial condition $f_1(\mathbf{r},\mathbf{v},t_o)$. If only waves that grow in time are considered, then $f_1(\mathbf{r},\mathbf{v},t_o)$ will go to zero as t_o ® ¥. Thus, the general first order perturbation to the velocity distribution function is

$$f_1(\vec{r}, \vec{v}, t) = -\frac{q}{M} \int_{-\infty}^{t} dt' \;\; [\vec{E}_1(\vec{r}\,', t') + \vec{v}\,' \times \vec{B}_1(\vec{r}\,', t')] \cdot \nabla_v f_o(\vec{r}\,', \vec{v}\,') \tag{8}$$

Since this section is only concerned about electrostatic waves, $\boldsymbol{k} \times \boldsymbol{B_1} = 0$, $\boldsymbol{B_1}$ can be set to zero.

$$f_1(\vec{r}, \vec{v}, t) = -\frac{q}{M} \int_{-\infty}^{t} dt' \;\; \vec{E}_1(\vec{r}\,', t') \cdot \nabla_v f_o(\vec{r}\,', \vec{v}\,') \tag{9}$$

To solve this equation, the first step is to rewrite the electric field as an electrostatic traveling wave.

$$\vec{E}_1(\vec{r},t) = \vec{E}_1 e^{i(\vec{k}\cdot\vec{r}-wt)} = -\nabla(\Phi e^{i(\vec{k}\cdot\vec{r}-wt)}) = -i\vec{k}\Phi e^{i(\vec{k}\cdot\vec{r}-wt)} \qquad (10)$$

where Φ is the amplitude of the electrostatic potential of the wave. Next a distribution function for equation (9) needs to be choosen. In a general case, any distribution function that is independent of time, i. e., a solution to the steady state zeroth order Vlasov equation can be used to compute $f_1(\boldsymbol{r},\boldsymbol{v},t)$. However, in general the most commonly used distribution function is the Bi-Maxwellian with a drift along the magnetic field as discribed by equation (11).

$$f_o(v_x,v_y,v_z) = \left(\frac{1}{2\pi\, v_{t||}}\right)^{1/2} e^{-(v_z-v_o)^2/2v_{t||}} \left(\frac{1}{2\pi\, v_{t\perp}}\right) e^{-(v_x^2+v_y^2)\;/2v_{t\perp}} \qquad (11)$$

Thus the equation for the first order distribution functions becomes

$$f(\vec{r},\vec{v},t) = -\frac{q}{M}\int_{-\infty}^{t} dt' i\vec{k}\Phi e^{i(\vec{k}\cdot\vec{r}'-wt')}\nabla_v \left(\frac{1}{2\pi\, v_{t\,||}}\right)^{1/2} e^{-(v_z'-v_o)^2/2v_{t\,||}} \times \left(\frac{1}{2\pi\, v_{t\perp}}\right) e^{-(v_x'^2+v_y'^2)/2v_{t\perp}} \qquad (12)$$

Next, integration over the unperturbed particle orbits is done. To complete this integral, the general unperturbed equations of motion are needed and can be derived from the motion of a charged particle in a uniform magnetic field with equation (13).

$$\frac{d\vec{v}'}{dt} = \vec{v}' \times \Omega \hat{e}_z \qquad (13)$$

where Ω is the cyclotron frequency and prime denotes variables that are functions of t' the variable for time. With t being the time of the initial conditions, τ is the time between the measerment and the initial conditions, $\tau = t' - t$. Solving these differential equations gives the unperturbed velocity equations which can be integrated to get the unperturbed equations for position.

Velocity equations	Position equations
$v_x' = v_x\cos(\Omega\tau) - \varepsilon v_y \sin(\Omega\tau)$	$x' = x - \frac{v_x}{\Omega}\sin(\Omega\tau) + \frac{\varepsilon v_y}{\Omega}(1-\cos(\Omega\tau))$
$v_y' = \varepsilon v_x \sin(\Omega\tau) + v_y\cos(\Omega\tau)$	$y' = y - \frac{\varepsilon v_x}{\Omega}(1-\cos(\Omega\tau)) - \frac{v_y}{\Omega}\sin(\Omega\tau)$
$v_z' = v_z$	$z' = z - v_z\tau$

Here e is $q/|q|$ where q is the charge. This provides the correct direction of rotation for electrons and ions. With these equations, equation (12) can be rewritten in terms of τ and the integration variable changed to τ. We start by rewriting the exponential term in terms of τ.

$$
\begin{aligned}
i(\vec{k}\cdot\vec{r}' - wt') = i(\vec{k}\cdot\vec{r} - wt) &+ \frac{iv_x}{\Omega}(-k_x \sin(\Omega t) - e k_y (1-\cos(\Omega t))) \\
&+ \frac{iv_y}{\Omega}(-k_y \sin(\Omega t) + e k_x (1-\cos(\Omega t))) \\
&+ i(w - k_z v_z)t
\end{aligned}
\tag{14}
$$

Going to cylindrical coordinates using

$$
\begin{aligned}
k_x &= k_\perp \cos f \\
k_y &= k_\perp \sin f
\end{aligned}
\tag{15}
$$

and $k_z = k_{||}$ the exponential in the electric field can be rewritten as

$$
\begin{aligned}
\exp[i(\vec{k}\cdot\vec{r}' - wt')] = \exp[&i(\vec{k}\cdot\vec{r} - wt) + i(\frac{ek_\perp}{\Omega})(v_y \cos f - v_x \sin f) \\
&+ i(\frac{k_\perp v_x}{\Omega})\sin(f - e\Omega t) + i(\frac{k_\perp v_y}{\Omega})\sin(f - e\Omega t - p/2) + i(w - k_{||} v_z)t\,]
\end{aligned}
\tag{16}
$$

Using the following Bessel identity

$$
e^{ib \sin q} = \sum_{n=-\infty}^{\infty} J_n(b) e^{inq}
\tag{17}
$$

the exponential term becomes

$$
\begin{aligned}
e^{i(\vec{k}\cdot\vec{r}' - wt')} = e^{i(\vec{k}\cdot\vec{r} - wt)} \cdot &\sum_{m,n=-\infty}^{\infty} J_m(\frac{k_\perp v_y}{\Omega}) J_n(\frac{k_\perp v_x}{\Omega}) \\
&\times e^{i(n+m)f} e^{i(w - e(n+m)\Omega - k_{||} v_z)t} e^{i\frac{ek_\perp}{\Omega}(v_y \cos f - v_x \sin f)} e^{-imp/2}
\end{aligned}
\tag{18}
$$

Now that the exponential term is written in terms of t, the gradient of the distribution function can also be rewritten.

$$
\vec{k}\cdot\nabla_{v'} f_o(\vec{v}') = -(\frac{v_x' k_\perp \cos f}{v_{t\perp}^2} + \frac{v_y' k_\perp \sin f}{v_{t\perp}^2} + \frac{(v_z' - v_o)k_z}{v_{t||}^2}) f_o(\vec{v}')
\tag{19}
$$

Making the appropriate substitutions into equation (19) for the primed variables yields

$$\vec{k}\cdot\nabla_{v'} f_o(\vec{v}') = -(\frac{(v_x\cos(\Omega t) - e v_y \sin(\Omega t))k_\perp \cos f}{v_{t\perp}^2} + \frac{(e v_x \sin(\Omega t) + v_y\cos(\Omega t))k_\perp \sin f}{v_{t\perp}^2} + \frac{(v_z - v_o)k_z}{v_{t\|}^2}) f_o(\vec{v}) \qquad (20)$$

which is equivalent to

$$\vec{k}\cdot\nabla_{v'} f_o(\vec{v}') = -(\frac{v_x k_\perp(\cos(\Omega t)\cos f + e\sin(\Omega t)\sin f)}{v_{t\perp}^2} + \frac{(e v_y k_\perp \sin(\Omega t)\cos f + \cos(\Omega t)\sin f)}{v_{t\perp}^2} + \frac{(v_z - v_o)k_z}{v_{t\|}^2}) f_o(\vec{v}) \qquad (21)$$

Using the trigonometric identities for the angle addition and subtraction equation (21) can be written as

$$\vec{k}\cdot\nabla_{v'} f_o(\vec{v}') = -\left(\frac{k_\perp}{v_{t\perp}^2}\left(v_x\cos(f - e\Omega t) + v_y\cos(f - e\Omega t + p/2)\right) + \frac{(v_z - v_o)k_z}{v_{t\|}^2}\right) f_o(\vec{v}) \qquad (22)$$

Equation (22) and equation (18) are subistituted into equation (12), and the integration variable changed to t using $t = t' - t$. The result is

$$f_1(\vec{r},\vec{v},t) = \frac{iq\Phi}{M} e^{i(\vec{k}\cdot\vec{r} - wt)} \int_{-\infty}^{0} dt \sum_{m,n=-\infty}^{\infty} J_m(\frac{k_\perp v_y}{\Omega}) J_n(\frac{k_\perp v_x}{\Omega}) \times e^{i(n+m)f - imp/2} e^{i(w - e(n+m)\Omega - k_\| v_z)t} e^{i\frac{ek_\perp}{\Omega}(v_y\cos f - v_x \sin f)} \times e^{-(\frac{k_\perp}{v_{t\perp}^2}\left(v_x\cos(f - e\Omega t) + v_y\cos(f - e\Omega t + p/2)\right)} + \frac{(v_z - v_o)k_\|}{v_{t\|}^2}) f_o(\vec{v}) \qquad (23)$$

Now the integral is a function of a single variable, t. The next step is to write the cosine terms in exponential form.

$$f_1(\vec{r},\vec{v},t) = \frac{iq\Phi}{M} e^{i(\vec{k}\cdot\vec{r} - wt)} e^{i\frac{ek_\perp}{\Omega}(v_y\cos f - v_x\sin f)} f_o(\vec{v}) \int_{-\infty}^{0} dt \sum_{m,n=-\infty}^{\infty} J_m(\frac{k_\perp v_y}{\Omega}) J_n(\frac{k_\perp v_x}{\Omega}) \times e^{i(w - e(n+m)\Omega - k_\| v_z)t + i(n+m)f - imp/2} \times\left(\frac{k_\perp v_x}{2v_{t\perp}^2}\left(e^{i(f - e\Omega t)} + e^{-i(f - e\Omega t)}\right) + \frac{k_\perp v_y}{2v_{t\perp}^2}\left(e^{i(f - e\Omega t + p/2)} + e^{-i(f - e\Omega t + p/2)}\right) + \frac{(v_z - v_o)k_\|}{v_{t\|}^2}\right) \qquad (24)$$

Rearranging equation (24) by multiplying each term through in the integral gives

$$f_1(\vec{r},\vec{v},t) = \frac{iq\Phi}{M} e^{i(\vec{k}\cdot\vec{r}-wt)} e^{i\frac{ek_\perp}{\Omega}(v_y\cos f - v_x \sin f)} f_o(\vec{v}) \int_{-\infty}^{0} dt \sum_{m,n=-\infty}^{\infty}$$

$$\times\left(\frac{k_\perp v_x}{2v_{t\perp}^2}\left(e^{i(w-e(n-1+m)\Omega-k_\parallel v_z)t+i(n+1+m)f} + e^{-i(w-e(n+1+m)\Omega-k_\parallel v_z)t+i(n-1+m)f}\right)e^{-imp/2} J_m(\frac{k_\perp v_y}{\Omega})J_n(\frac{k_\perp v_x}{\Omega}) + \right.$$

$$\frac{k_\perp v_y}{2v_{t\perp}^2}\left(e^{i(w-e(n+m-1)\Omega-k_\parallel v_z)t+i(n+m-1)f-i(m-1)p/2} + e^{-i(w-e(n+m+1)\Omega-k_\parallel v_z)t+i(n+m-1)f-i(m+1)p/2}\right) J_m(\frac{k_\perp v_y}{\Omega})J_n(\frac{k_\perp v_x}{\Omega}) + \qquad (25)$$

$$\left.\frac{(v_z-v_o)k_z}{v_{t\parallel}^2} e^{i(w-e(n+m)\Omega-k_\parallel v_z)t+i(n+m)f} e^{-imp/2} J_m(\frac{k_\perp v_y}{\Omega})J_n(\frac{k_\perp v_x}{\Omega})\right)$$

Next the indices of the summations are adjusted to reduce equation (25). For simplicity lets work with one term as an example.

$$\frac{k_\perp v_y}{2v_{t\perp}^2}\sum_{n,m=-\infty}^{\infty} e^{-i(w-e(n+m+1)\Omega-k_\parallel v_z)t+i(n+m-1)f-i(m+1)p/2} J_m(\frac{k_\perp v_y}{\Omega})J_n(\frac{k_\perp v_x}{\Omega}) \qquad (26)$$

Now let $p=m-1$, or $m=p+1$.

$$\frac{k_\perp v_y}{2v_{t\perp}^2}\sum_{n,(p+1)=-\infty}^{\infty} e^{-i(w-e(n+p)\Omega-k_\parallel v_z)t+i(n+p)f-i(p)p/2} J_{p+1}(\frac{k_\perp v_y}{\Omega})J_n(\frac{k_\perp v_x}{\Omega}) \qquad (27)$$

Since the summation is from $-\infty$ to $+\infty$, and the terms get small as they go to infinity, the result of equation (27) is the same as

$$\frac{k_\perp v_y}{2v_{t\perp}^2}\sum_{n,p=-\infty}^{\infty} e^{-i(w-e(n+p)\Omega-k_\parallel v_z)t+i(n+p)f+i(p)p/2} J_{p+1}(\frac{k_\perp v_y}{\Omega})J_n(\frac{k_\perp v_x}{\Omega}) \qquad (28)$$

Now, even though it is confusing, p is set equal to m $(p=m)$ to give

$$\frac{k_\perp v_y}{2v_{t\perp}^2}\sum_{n,m=-\infty}^{\infty} e^{-i(w-e(n+m)\Omega-k_\parallel v_z)t+i(n+m)f+i(m)p/2} J_{m+1}(\frac{k_\perp v_y}{\Omega})J_n(\frac{k_\perp v_x}{\Omega}) \qquad (29)$$

Applying this technique to the necessary terms, equation (25) can be written as

$$f_1(\vec{r},\vec{v},t) = \frac{iq\Phi}{M} e^{i(\vec{k}\cdot\vec{r}-wt)} e^{i\frac{ek_\perp}{\Omega}(v_y\cos f - v_x \sin f)} f_o(\vec{v}) \int_{-\infty}^{0} dt \sum_{m,n=-\infty}^{\infty}$$

$$\times\left(\frac{k_\perp v_x}{2v_{t\perp}^2}\left(J_{n-1}(\frac{k_\perp v_x}{\Omega}) + J_{n+1}(\frac{k_\perp v_x}{\Omega})\right)J_m(\frac{k_\perp v_y}{\Omega}) \right.$$

$$+\frac{k_\perp v_y}{2v_{t\perp}^2}\left(J_{m-1}(\frac{k_\perp v_y}{\Omega}) + J_{m+1}(\frac{k_\perp v_y}{\Omega})\right)J_n(\frac{k_\perp v_x}{\Omega}) \qquad (30)$$

$$\left. + \frac{(v_z-v_o)k_z}{v_{t\parallel}^2} J_m(\frac{k_\perp v_y}{\Omega})J_n(\frac{k_\perp v_x}{\Omega})\right) e^{i(w-e(n+m)\Omega-k_\parallel v_z)t+(n+m)f-imp/2}$$

Now the Bessel relationship,

$$J_n(x) = \frac{x}{2n}\left(J_{n-1}(\frac{k_\perp v_x}{\Omega}) + J_{n+1}(\frac{k_\perp v_x}{\Omega}) \right) \tag{31}$$

can be used to rewrite equation (30) as

$$\begin{aligned} f_1(\vec{r},\vec{v},t) &= \frac{iq\Phi}{M} e^{i(\vec{k}\cdot\vec{r} - w\ t)} e^{i\frac{ek_\perp}{\Omega}(v_y \cos f - v_x \sin f)} f_o(\vec{v}) \\ &\times \sum_{m,n=-\infty}^{\infty} (\frac{n\Omega}{v_{t\perp}^2} + \frac{m\Omega}{v_{t\perp}^2} + \frac{(v_z - v_o)k_\parallel}{v_{t\parallel}^2}) J_m(\frac{k_\perp v_y}{\Omega}) J_n(\frac{k_\perp v_x}{\Omega}) \\ &\times e^{+i(n+m)f - imp/2} \int_{-\infty}^{0} dt\ \ e^{i(w - e(n+m)\Omega - k_\parallel v_z)t} \end{aligned} \tag{32}$$

To do the integral over τ, it is assumed that the frequency, ω, has a small but finite imaginary part, $w = w_{real} + iw_{imaginary}$. This makes it possible to do the integral in equation (32) since it goes to zero in the limit of ∞. Physically this means the waves have a finite growth rate since they have been assumed to grow in time. Thus the integral is

$$\int_{-\infty}^{0} dt\ \ e^{i(w - e(n+m)\Omega - k_\parallel v_z)t} = \frac{1}{i(w - e(n+m)\Omega - k_\parallel v_z)} \tag{33}$$

Putting this back into equation (32) gives

$$\begin{aligned} f_1(\vec{r},\vec{v},t) &= (\frac{-e\Phi}{M}) e^{-i(\vec{k}\cdot\vec{r} - wt)} f_o(\vec{v}) \sum_{m,n} J_m(\frac{k_\perp v_y}{\Omega})\quad J_n(\frac{k_\perp v_x}{\Omega}) \\ &\times e^{-imp/2} e^{i(m+n)f} e^{ie(k_x v_y/\Omega - k_y v_x/\Omega)} \\ &\times \left(\frac{\frac{e\Omega(m+n)}{v_{t\perp}^2} + \frac{k_\parallel(v_z - v_o)}{v_{t\parallel}^2}}{e(n+m)\Omega + k_\parallel v_z - w} \right) \end{aligned} \tag{34}$$

Equation (34) is the perturbed velocity distribution function and can be found in Safarty *et. al.*[7] with the addition of the *exp(-i(**k**· **r**-wt))* term. Since the perturbed velocity distribution function is created by a wave, the phase of the perturbation depends on the phase of the wave. Thus, if you measure $f_1(\mathbf{r},\mathbf{v},t)$ at two different positions or times in a wave the phase will be different, and this phase difference propagates as *exp(-i(**k**· **r**-wt))* just as the wave. This term could be removed and the remaining part of $f_1(\mathbf{r},\mathbf{v},t)$ thought of as a Fourier amplitude such that $f_1(\mathbf{r},\mathbf{v},t) = exp(-i(\mathbf{k}\cdot \mathbf{r}-wt))|\, f_1(\mathbf{v})|$, but will be left in for this development.

2.1 Integrating over velocities

The purpose of this calculation is to find an equation that can be used to fit experimental LIF data. Since LIF measures only one component of velocity, the equation for $f_1(\mathbf{r},\mathbf{v},t)$ must be a function of a single component of velocity. To achieve this, integration over two components of velocity can be done to get a function of in terms of the single remaining component. The velocity components in equation (34) can be separated as follows

$$f_1(\vec{r},\vec{v},t) = (\frac{-e\Phi}{M})\, e^{-i(\vec{k}\cdot\vec{r}-wt)} \sum_{m,n} e^{-imp/2} e^{i(m+n)f}$$
$$\times\left[\left(\frac{1}{2p\, v_{t\perp}^2}\right)^{1/2} e^{-v_y^2/2v_{t\perp}} e^{iek_x v_y/\Omega} J_m(\frac{k_\perp v_y}{\Omega})\right]$$
$$\times\left[\left(\frac{1}{2p\, v_{t\perp}^2}\right)^{1/2} e^{-v_x^2/2v_{t\perp}} e^{-iek_y v_x/\Omega} J_n(\frac{k_\perp v_x}{\Omega})\right] \qquad (35)$$

$$\times\left[\left(\frac{1}{2p\, v_{t\|}^2}\right)^{1/2} e^{-(v_z-v_o)^2/2v_{t\|}} \left(\frac{\frac{e\Omega(m+n)}{v_{t\perp}^2}+\frac{k_\|(v_z-v_o)}{v_{t\|}^2}}{e\,(n+m)\Omega+k_\| v_z - w}\right)\right]$$

where each term in brakets is a function of only one compenent of velocity. Integrating over any two of the terms in brakets with respect to the velocity variable in the braket will leave behind $f_1(\mathbf{r},\mathbf{v},t)$ as a function of the remaining variable. The next thing to do is to integrate each of the terms in brakets. Then we can substitute any two of the results for the terms in brakets to get a function of the remaining variable.

2.1.1 Integrating over v_z

The integral of $f_1(\mathbf{r},v,t)$ over v_z, is as follows:

$$\left(\frac{1}{2pv_{t\|}^2}\right)^{1/2} \int_{-\infty}^{\infty} dv_z \left(\frac{1}{2pv_{t\|}^2}\right)^{1/2} e^{-(v_z-v_o)^2/2v_{t\|}} \left(\frac{\frac{e\Omega(m+n)}{v_{t\perp}^2}+\frac{k_\|(v_z-v_o)}{v_{t\|}^2}}{e\,(n+m)\Omega+k_\| v_z - w}\right) \qquad (36)$$

To do this integral, the following substitution is used

$$s = \frac{v_z - v_o}{\sqrt{2}\;\; v_{t\|}} \qquad (37)$$
$$dv_z = \sqrt{2}\;\; v_{t\|} ds$$

Using these relationships, the integral can be rewritten as

$$\left(\frac{1}{2p\, v_{t||}^2}\right)^{1/2} \int_{-\infty}^{\infty} \sqrt{2}\quad v_{t||} ds\ e^{-s^2} \left(\frac{\dfrac{e\Omega(m+n)}{v_{t\perp}^2} + \dfrac{\sqrt{2}\, k_{||} s}{v_{t||}}}{\sqrt{2}\, k_{||} v_{t||}\left(s + \dfrac{e(n+m)\Omega + k_{||} v_o - w}{\sqrt{2}\, k_{||} v_{t||}} \right)} \right) \tag{38}$$

Let

$$d_{n+m} = \frac{w - e(n+m)\Omega - k_{||} v_o}{\sqrt{2}\, k_{||} v_{t||}} \tag{39}$$

and equation (38) can be rewritten as

$$\frac{1}{\sqrt{p}} \int_{-\infty}^{\infty} ds \ \left(\frac{e\Omega(m+n)}{v_{th\perp}^2 \sqrt{2}\, k_z v_{t||}} + \frac{s}{v_{t||}^2} \right) \frac{e^{-s^2}}{s - d_{m+n}} \tag{40}$$

Using the fact that $d_0=(k_{||}v_z\text{-}w)/Ö2\ \overline{k}_z v_{t||}$ equation (40) can be written as

$$\frac{1}{\sqrt{p}} \int_{-\infty}^{\infty} ds \left(\frac{d_0 - d_{n+m}}{v_{th\perp}^2} + \frac{s}{v_{t||}^2} \right) \frac{e^{-s^2}}{s + d_{m+n}} \tag{41}$$

Now, equation (41) is in a form that can be integrated with the help of the plasma dispersion relationships

$$Z(d) = \frac{1}{\sqrt{p}} \int_{-\infty}^{\infty} ds \frac{e^{-s^2}}{s - d} \tag{42}$$

$$Z'(d) = \frac{-2}{\sqrt{p}} \int_{-\infty}^{\infty} ds \frac{s e^{-s^2}}{s - d} \tag{43}$$

$$Z'(d) = -2[1 + d Z(d)] \tag{44}$$

Using these relationships, equation (41) can be written as

$$v_{t||}^2 \left[\frac{v_{t||}^2}{v_{t\perp}^2} (d_0 - d_{n+m}) Z(d_{n+m}) + \frac{1}{2} Z'(d_{n+m}) \right] \tag{45}$$

$$v_{t||}^2 \left[\frac{v_{t||}^2}{v_{t\perp}^2} (d_0 - d_{n+m}) Z(d_{n+m}) + 1 + d_{n+m} Z(d_{n+m}) \right] \tag{46}$$

This can be reduced further using the fact that $v_{t||}{}^2/v_{t\wedge}{}^2 = T_{||}/T_\wedge$:

$$\int_{-\infty}^{\infty} dv_z \left(\frac{1}{2p v_{t||}^2}\right)^{1/2} e^{-(v_z-v_o)^2/2v_{t||}} \left(\frac{\dfrac{e\Omega(m+n)}{v_{t\perp}^2} + \dfrac{k_{||}(v_z - v_o)}{v_{t||}^2}}{e(n+m)\Omega + k_{||} v_z - w} \right) = \tag{47}$$

$$v_{t||}^2 \left[1 + \left(\frac{T_{||}}{T_\perp} d_0 \left(1 + \frac{T_{||}}{T_\perp} \right) d_{n+m} \right) Z(d_{n+m}) \right]$$

2.1.2 Integrating over $\mathbf{v_x}$

The integral over v_x is

$$\int_{-\infty}^{\infty} dv_x \left(\frac{1}{2p v_{t\perp}^2}\right)^{1/2} e^{-v_x^2/2v_{t\perp}} e^{-iek_y v_x/\Omega} J_n\left(\frac{k_\perp v_x}{\Omega}\right) \tag{48}$$

First the ordinary Bessel function can be rewritten as

$$J_n\left(\frac{k_\perp v_x}{\Omega}\right) = \frac{1}{2p} \int_0^{2p} da \; e^{i\frac{k_\perp v_x}{\Omega}\sin a - ina} \tag{49}$$

to make the integral easier. With equation (49), equation (48) can be written as

$$\left(\frac{1}{2p v_{t\perp}^2}\right)^{1/2} \int_{-\infty}^{\infty} dv_x e^{-v_x^2/2v_{t\perp}} e^{-iek_y v_x/\Omega} \frac{1}{2p} \int_0^{2p} da \quad e^{i\frac{k_\perp v_x}{\Omega}\sin a - ina} \tag{50}$$

$$\left(\frac{1}{2p v_{t\perp}^2}\right)^{1/2} \frac{1}{2p} \int_0^{2p} da \quad e^{-ina} \int_{-\infty}^{\infty} dv_x e^{i\frac{k_\perp v_x}{\Omega}(\sin a - e\sin f) - \frac{v_x^2}{2v_{t\perp}^2}} \tag{51}$$

The next step in integrating over v_x is to complete the square of the exponential term.

$$\left(\frac{1}{2p v_{t\perp}^2}\right)^{1/2} \int_{-\infty}^{\infty} dv_x e^{-i\frac{k_\perp v_x}{\Omega}(e\sin f - \sin a) - \frac{v_x^2}{2v_{t\perp}^2} - \frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}(e\sin f - \sin a)^2 + \frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}(e\sin f - \sin a)^2} \tag{52}$$

$$\left(\frac{1}{2p v_{t\perp}^2}\right)^{1/2} \int_{-\infty}^{\infty} dv_x e^{-\left[\frac{v_x}{\sqrt{2}v_{t\perp}} + i\frac{k_\perp v_{t\perp}}{\sqrt{2}\Omega}(e\sin f - \sin a)\right]^2 - \frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}(e\sin f - \sin a)^2} \tag{53}$$

U substitution is used to complete the integral with

$$u = \frac{v_x}{\sqrt{2}v_{t\perp}} + i\frac{k_\perp v_{t\perp}}{\sqrt{2}\,\Omega}(e\ \sin f - \sin a\,)$$
$$dv_x = \sqrt{2}v_{t\perp}du \tag{54}$$

Now equation (53) can be written as

$$\left(\frac{1}{2p\,v_{t\perp}^2}\right)^{1/2} e^{-\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}(\sin a - \sin f\,)^2} \sqrt{2}v_{t\perp}\int_{-\infty}^{\infty} du\ \ e^{-u^2} \tag{55}$$

since the last term in the exponential has no v_x dependence. With the evaluation of the integral

$$\int_{-\infty}^{\infty} du\, e^{-u^2} = \sqrt{p} \tag{56}$$

equation (53) becomes

$$\left(\frac{1}{2p\,v_{t\perp}^2}\right)^{1/2} \int_{-\infty}^{\infty} dv_x e^{-\left[\frac{v_x}{\sqrt{2}v_{t\perp}} + i\frac{k_\perp v_{t\perp}}{\sqrt{2}\,\Omega}(e\sin f - \sin a\,)\right]^2 - \frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}(e\sin f - \sin a\,)^2} = e^{-\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}(e\sin f - \sin a\,)^2} \tag{57}$$

Substituting back into equation (51) gives

$$\frac{1}{2p}\int_0^{2p} da\quad e^{-ina}e^{-\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}(e\sin f - \sin a\,)^2} \tag{58}$$

The next step is to expand and algebraically transform the exponential term with the following identity

$$e^{\,l\cos\Omega t} = \sum_{n=-\infty}^{\infty} I_n(\,l\,)e^{in\Omega t} \tag{59}$$

The next set of steps transforms equation (58) into a form for use with equation (59).

$$\frac{1}{2p}\int_0^{2p} da\quad e^{-ina}e^{-\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}(\sin^2 f - 2\,e\sin f\sin a + \sin^2 a\,)} \tag{60}$$

$$\frac{1}{2p}e^{-\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}\sin^2 f}\int_0^{2p} da\ e^{-ina}\ e^{-\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}\sin^2 a + \frac{k_\perp^2 v_{t\perp}^2}{\Omega^2}e\sin f\sin a} \tag{61}$$

Using some trigonometric identities yields

$$\frac{1}{2p} e^{-\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}\sin^2 f} \int_0^{2p} da \quad e^{-ina} e^{-\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}(\frac{1}{2}-\frac{1}{2}\cos 2a)+\frac{k_\perp^2 v_{t\perp}^2}{\Omega^2} e \sin f \cos(a+\frac{p}{2})} \tag{62}$$

Using the identity of equation (59), equation (62) becomes

$$\frac{1}{2p} e^{-\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}\sin^2 f} e^{-\frac{k_\perp^2 v_{t\perp}^2}{4\Omega^2}} \int_0^{2p} da\, e^{-ina} \sum_{2l=-\infty}^{\infty} I_l\left(\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}\right) e^{i2la} \sum_{p=-\infty}^{\infty} I_p\left(\frac{e k_\perp^2 v_{t\perp}^2 \sin f}{\Omega^2}\right) e^{ipa+ip\frac{p}{2}} \tag{63}$$

$$\frac{1}{2p} e^{-\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}\sin^2 f} e^{-\frac{k_\perp^2 v_{t\perp}^2}{4\Omega^2}} \sum_{p=-\infty}^{\infty} I_p\left(\frac{e k_\perp^2 v_{t\perp}^2 \sin f}{\Omega^2}\right) e^{ip\frac{p}{2}} \sum_{l=-\infty}^{\infty} I_l\left(\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}\right) \int_0^{2p} da \quad e^{-ina+i2la+ipa} \tag{64}$$

The integral over α is completed using orthogonality

$$\int_0^{2p} da\ e^{-ina+i2la+ipa} = \begin{cases} 2p & for\ \ 2l+p-n=0 \\ 0 & for\ \ 2l+p-n\neq 0 \end{cases} \tag{65}$$

giving $l = (n-p)/2$ where l must be an integer. Using this, equation (64) becomes

$$e^{-\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}\sin^2 f} e^{-\frac{k_\perp^2 v_{t\perp}^2}{4\Omega^2}} \sum_{p=-\infty}^{\infty} I_p\left(\frac{e k_\perp^2 v_{t\perp}^2 \sin f}{\Omega^2}\right) I_{\frac{n-p}{2}}\left(\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}\right) e^{ip\frac{p}{2}} \tag{66}$$

If we define *a=Ö2k̄^p (a=Ö2k̄∧v_t∧/W)* and *c=Ö2k̄∧v_t∧sinf*, this can be written as

$$e^{-\left(\frac{c^2}{4}+\frac{a^2}{8}\right)} \sum_{p=-\infty}^{\infty} I_p\left(\frac{eac}{2}\right) I_{\frac{n-p}{2}}\left(\frac{a^2}{8}\right) e^{ip\frac{p}{2}} \tag{67}$$

Thus the integral over v_x is

$$\int_{-\infty}^{\infty} dv_x \left(\frac{1}{2p v_{t\perp}^2}\right)^{1/2} e^{-v_x^2/2v_{t\perp}} e^{-iek_y v_x/\Omega} J_n\left(\frac{k_\perp v_x}{\Omega}\right) = e^{-\left(\frac{c^2}{4}+\frac{a^2}{8}\right)} \sum_{p=-\infty}^{\infty} I_p\left(\frac{eac}{2}\right) I_{\frac{n-p}{2}}\left(\frac{a^2}{8}\right) e^{ip\frac{p}{2}} \tag{68}$$

2.1.3 Integrating over v_y

The integral over v_y is nearly the same as that over v_x, but to be explicit the integration is also done here. The integral over v_y is

$$\int_{-\infty}^{\infty} dv_y \left(\frac{1}{2p\, v_{t\perp}^2}\right)^{1/2} e^{-v_y^2/2v_{t\perp}} e^{iek_x v_y/\Omega} J_m\left(\frac{k_\perp v_y}{\Omega}\right)$$

(69)

As before ordinary Bessel function is written as

$$J_m\left(\frac{k_\perp v_y}{\Omega}\right) = \frac{1}{2p}\int_0^{2p} da \;\; e^{i\frac{k_\perp v_y}{\Omega}\sin a - ima} \tag{70}$$

to make the integral easier. With this relationship equation (69) can be written as

$$\left(\frac{1}{2p v_{t\perp}^2}\right)^{1/2} \int_{-\infty}^{\infty} dv_y e^{-v_y^2/2v_{t\perp}} e^{iek_x v_y/\Omega} \frac{1}{2p}\int_0^{2p} da \;\; e^{i\frac{k_\perp v_y}{\Omega}\sin a - ima} \tag{71}$$

$$\left(\frac{1}{2p v_{t\perp}^2}\right)^{1/2} \frac{1}{2p}\int_0^{2p} da\, e^{-ima} \int_{-\infty}^{\infty} dv_y e^{i\frac{k_\perp v_y}{\Omega}(e\cos f + \sin a) - \frac{v_y^2}{2v_{t\perp}^2}} \tag{72}$$

The next step in the integral over v_y is to complete the square of the exponential term.

$$\left(\frac{1}{2p\, v_{t\perp}^2}\right)^{1/2} \int_{-\infty}^{\infty} dv_y e^{i\frac{k_\perp v_y}{\Omega}(e\cos f + \sin a) - \frac{v_y^2}{2v_{t\perp}^2} - \frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}(e\cos f + \sin a)^2 + \frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}(e\cos f + \sin a)^2} \tag{73}$$

$$\left(\frac{1}{2p\, v_{t\perp}^2}\right)^{1/2} \int_{-\infty}^{\infty} dv_y e^{-\left[\frac{v_y}{\sqrt{2}v_{t\perp}} - i\frac{k_\perp v_{t\perp}}{\sqrt{2}\Omega}(e\cos f + \sin a)\right]^2 - \frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}(e\cos f + \sin a)^2} \tag{74}$$

U substitution is used to complete the integral with

$$u = \frac{v_y}{\sqrt{2}v_{t\perp}} + i\frac{k_\perp v_{t\perp}}{\sqrt{2}\Omega}(e\cos f + \sin a)$$
$$dv_x = \sqrt{2}v_{t\perp}du \tag{75}$$

Now equation (74) can be written as

$$\left(\frac{1}{2p v_{t\perp}^2}\right)^{1/2} e^{\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}(e\cos f + \sin a)^2} \sqrt{2}v_{t\perp}\int_{-\infty}^{\infty} du \;\; e^{-u^2} \tag{76}$$

since the last term in the exponential has no v_y dependence. With the evaluation of the integral

$$\int_{-\infty}^{\infty} du\, e^{-u^2} = \sqrt{p} \tag{77}$$

equation (74) becomes

$$\left(\frac{1}{2p v_{t\perp}^2}\right)^{1/2} \int_{-\infty}^{\infty} dv_y e^{-\left[\frac{v_y}{\sqrt{2}v_{t\perp}} + i\frac{k_\perp v_{t\perp}}{\sqrt{2}\Omega}(e\cos f + \sin a)\right]^2 + \frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}(e\cos f + \sin a)^2} = e^{\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}(e\cos f + \sin a)^2}$$

(78)

Substituting back into equation (72) we get

$$\frac{1}{2p}\int_0^{2p} da \quad e^{-ima} e^{\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}(e\cos f+\sin a)^2} \tag{79}$$

The next step is to expand and algebraically transform the exponential term using the following identity

$$e^{l\cos\Omega t} = \sum_{n=-\infty}^{\infty} I_n(l)e^{in\Omega t} \tag{80}$$

The next set of steps transforms equation (79) into a form for use with equation (80).

$$\frac{1}{2p}\int_0^{2p} da \quad e^{-ima} e^{\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}(\cos^2 f+2e\cos f\,\sin a+\sin^2 a)} \tag{81}$$

$$\frac{1}{2p} e^{\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}\cos^2 f}\int_0^{2p} da \quad e^{-ima} e^{\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}\sin^2 a+\frac{k_\perp^2 v_{t\perp}^2}{\Omega^2}e\cos f\,\sin a} \tag{82}$$

Using some trigonometric identities yields

$$\frac{1}{2p} e^{\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}\cos^2 f}\int_0^{2p} da \quad e^{-ima} e^{\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}(\frac{1}{2}-\frac{1}{2}\cos 2a)+\frac{k_\perp^2 v_{t\perp}^2}{\Omega^2}e\cos f\cos(a+\frac{p}{2})} \tag{83}$$

Now using equation (80), equation (83) becomes

$$\frac{1}{2p} e^{\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}\cos^2 f} e^{\frac{k_\perp^2 v_{t\perp}^2}{4\Omega^2}}\int_0^{2p} da \quad e^{-ima}\sum_{2l=-\infty}^{\infty} I_l\left(\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}\right)e^{-i2la}\sum_{p=-\infty}^{\infty} I_p\left(\frac{ek_\perp^2 v_{t\perp}^2\cos f}{\Omega^2}\right)e^{ipa+ip\frac{p}{2}} \tag{84}$$

$$\frac{1}{2p} e^{\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}\cos^2 f} e^{\frac{k_\perp^2 v_{t\perp}^2}{4\Omega^2}}\sum_{p=-\infty}^{\infty} I_p\left(\frac{ek_\perp^2 v_{t\perp}^2\cos f}{\Omega^2}\right)e^{ip\frac{p}{2}}\sum_{l=-\infty}^{\infty} I_l\left(\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}\right)\int_0^{2p} da \quad e^{-ima-i2la+ipa} \tag{85}$$

The integral over α is done using orthogonality

$$\int_0^{2p} da \quad e^{-ima+i2la+ipa} = \begin{cases} 2p & for \quad -2l+p-m=0 \\ 0 & for \quad -2l+p-m\neq 0 \end{cases} \tag{86}$$

gives *l = (p-m)/2* where l must be an integer. Thus, equation (85) becomes

$$e^{\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}\cos^2 f} e^{\frac{k_\perp^2 v_{t\perp}^2}{4\Omega^2}}\sum_{p=-\infty}^{\infty} I_p\left(\frac{ek_\perp^2 v_{t\perp}^2\cos f}{\Omega^2}\right)I_{\frac{p-m}{2}}\left(\frac{k_\perp^2 v_{t\perp}^2}{2\Omega^2}\right)e^{ip\frac{p}{2}}$$

(87)

If we define $a=Ö2\bar{k}\,{}^\wedge p$ $(a=Ö2\bar{k}_\wedge v_{t\wedge}/W)$ and $d=Ö2\bar{k}_\wedge v_{t\wedge}cos f$, this can be written as

$$e^{-\left(\frac{d^2}{4}-\frac{a^2}{8}\right)}\sum_{p=-\infty}^{\infty} I_p\left(\frac{e\,ad}{2}\right)I_{\frac{p-m}{2}}\left(\frac{a^2}{8}\right)e^{ip\frac{p}{2}} \tag{88}$$

Thus the integral over v_y is

$$\int_{-\infty}^{\infty} dv_y\left(\frac{1}{2pv_{t\perp}^2}\right)^{1/2} e^{-v_y^2/2v_{t\perp}^2}e^{iek_y v_y/\Omega}J_m\left(\frac{k_\perp v_y}{\Omega}\right)=e^{-\left(\frac{d^2}{4}-\frac{a^2}{8}\right)}\sum_{p=-\infty}^{\infty} I_p\left(\frac{ead}{2}\right)I_{\frac{p-m}{2}}\left(\frac{a^2}{8}\right)e^{ip\frac{p}{2}} \tag{89}$$

2.1.4 Single Velocity Component Electrostatic f_1(r,v,t)

With the integrals over each of the individual velocity components completed, the results can be substituted back into equation (35) to get $f_1(\mathbf{r},\mathbf{v},t)$ as a function of a sing velocity component. Since LIF is typically done in the perpendicular or parallel direction relative to the background magnetic field, this suggests that $f_1(v)$ should be a function of v_z and v_x or v_y. As for v_x and v_y, only one direction is needed because $f_1(\mathbf{r},v_y,t)$ as a function of one or the other is equivalent due to the symmetry of the particle motion. Since this motion is symmetric along the z axis, the coordinate system can be rotated around the z axis making $f_1(\mathbf{r},v_y,t)$ or $f_1(\mathbf{r},v_x,r)$ sufficient for perpendicular measurements. Below is $f_1(\mathbf{r},v_y,t)$, equation (90), the same as Safarty *et al*.[7]

$$\begin{aligned} f_1(\vec{r},v_y,t)=&\left(\frac{-e\Phi}{p\,M\,v_{th\|}^{\;2}}\right)e^{-i(\vec{k}\cdot\vec{r}-wt)}\,f_o(v_y)\sum_{n,m}\left(\left(1+Z(z_{n+m})\right)\left(z_0\frac{T_\|}{T_\perp}+\left(1-\frac{T_\|}{T_\perp}\right)z_{n+m}\right)\right)\\ &\times J_m\left(\frac{k_\perp v_y}{\Omega}\right)e^{-imp/2}e^{i(m+n)q}\,e^{iek_x v_y/\Omega}e^{-a^2/8}e^{-c^2/4}\\ &\times\sum_p I_{(n-p)/2}\left(\frac{a^2}{8}\right)I_p\left(\frac{eac}{2}\right)e^{-ipp/2} \end{aligned} \tag{90}$$

With the change of sumation indices in equation (68) to avoid confusion of terms, equation (91) is $f_1(v_z)$.

$$\begin{aligned} f_1(\vec{r},v_z,t)=&\left(\frac{-e\Phi}{p\;M\;v_{th\|}^{\;2}}\right)e^{-i(\vec{k}\cdot\vec{r}-wt)}\,f_o(v_z)\sum_{m,n}e^{-imp/2}e^{i(m+n)f}\left(\frac{\frac{e\Omega(m+n)}{v_{t\perp}^2}+\frac{k_\|(v_z-v_o)}{v_{t\|}^2}}{e(n+m)\Omega+k_\| v_z-w}\right)\\ &\times e^{-d^2/4}\sum_{p=-\infty}^{\infty} I_p\left(\frac{e\,ad}{2}\right)I_{\frac{m-p}{2}}\left(\frac{a^2}{8}\right)e^{-ip\frac{p}{2}}\\ &\times e^{-c^2/4}\sum_{l=-\infty}^{\infty} I_l\left(\frac{e\,ac}{2}\right)I_{\frac{n-l}{2}}\left(\frac{a^2}{8}\right)e^{-il\frac{p}{2}} \end{aligned} \tag{91}$$

2.2 Constraint of w, $k_\wedge$, and $k_{||}$ using the Dispersion Relationship

As with any wave, there is a dispersion relationship defines the relationship between the frquency, w, and the wave number, k. In this case, electrostatic plasma waves have been assumed, so the dispersion relation for electostatic waves is used, equation (92). This equation relates w, $k_\wedge$, and $k_{||}$, or provides an additional constraint. This allows the elimination of another variable from equation (90) or (91) and imposes the wave relationship that pertrubs the velocity distribution. Selecting the perturbing wave is the most important part of the constraint because it is the wave length of this wave that the $f_1(\mathbf{r},v,t)$ measurement is attempting to deteremine. Thus, this information should be contained in the process somewhere. So for electrostatic waves, the dispersion relation is

$$k^2 e(k_{\|},k_{\perp},w) = k_{||}^{\ 2} + k_{\perp}^{\ 2} + \sum_{i,e} k_{ds}^{\ 2}\left(1+\sum_n e^{-a_s} I_n(a_s) \times Z(z_n)(z_n + \frac{n\Omega T_{||}}{\sqrt{2}k_{\|}v_{th||}T_{\perp}})\right) = 0 \tag{92}$$

where k_{ds} is the debye wave number, the sum over i and e are for electrons and ions. All other parameters are the same as before. Using all of this information, the theoretical real and imaginary parts of $f_1(\mathbf{r},v_y,t)$ can be generated for electrostatic ion cyclotron waves. The theoretical real and imaginary part for two different $f_1(\mathbf{r},v_y,t)s$, using equations (90) and (92), are shown in Figure 1 and Figure 2 for different sets of plasma parameters. One important thing to note here is that there a term that causes the phase as a function of velocity. It is the $\exp(ik_x v_y/W)$ term.

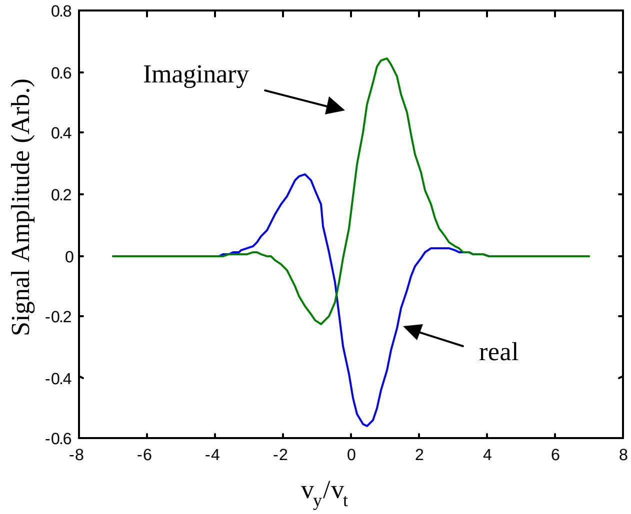

Figure 1: Real and Imaginary parts of an $f_1(\mathbf{r},v_y,t)$ for an argon plasma confined by a 420 gauss magnetic field, with approximately a 10,000 cm/s drift in the z direction, and an isotropic temperature of 0.1 eV. The wave numbers for the ion cylcotron wave are $k_{\wedge}$ » 1.5 cm^{-1} and $k_{||}$ » 0.58 cm^{-1}.

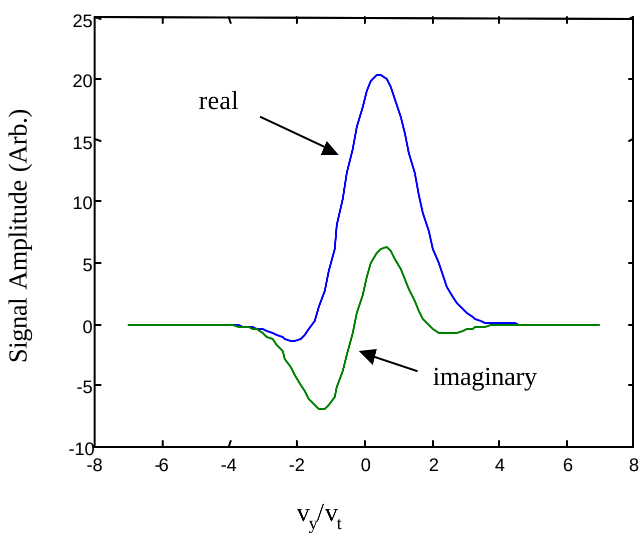

Figure 2: Real and Imaginary parts of an $f_1(\mathbf{r},v_y,t)$ for an argon plasma with a 1000 gauss magnetic field, no drift in the z direction, and an anisotropic temperature of $T_{||}/T_{\wedge} = 2$. The wave numbers for the ion cyclotron wave are $k_{\wedge}$ » 3.82 cm^{-1} and $k_{||}$ » 0.62 cm^{-1}.

3 ElectroMagnetic f_1(r,v,t)

The measurement of $f_1(\mathbf{r},\mathbf{v},t)$ can be generalized to include electromagnetic waves as well. This means a theoretical calculation for a full electromagnetic $f_1(\mathbf{r},\mathbf{v},t)$ need to be completed. To do this, Faraday's law can be used to relate the electric field to the magnetic field.

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \tag{93}$$

Again using traveling waves the magnetic field **B** can be written in terms of the electric field **E**.

$$i\vec{k}\times\vec{E}=iw\vec{B} \tag{94}$$

$$\vec{B}=\frac{\vec{k}\times\vec{E}}{w} \tag{95}$$

Substituting this into equation (8) from section 2.0, the electromagnetic $f_1(\mathbf{r},\mathbf{v},t_o)$ is

$$f_1(\vec{r},\vec{v},t)=-\frac{q}{M}\int_{-\infty}^{t}dt'\ [\vec{E}(\vec{r}',t')+v'\times\frac{\vec{k}\times\vec{E}(\vec{r}',t')}{w}]\cdot\nabla_v f_o(\vec{r}',\vec{v}') \tag{96}$$

Substituting in the traveling wave solution into equation (96) produces

$$f_1(\vec{r},\vec{v},t)=-\frac{q}{M}\int_{-\infty}^{t}dt'\ [\vec{E}+\vec{v}'\times\frac{\vec{k}\times\vec{E}}{w}]\cdot\nabla_v f_o(\vec{v}')\cdot e^{i(\vec{k}\cdot\vec{r}'-wt')} \tag{97}$$

Now using vector identities equation (97) can be written as

$$f_1(\vec{r},\vec{v},t)=-\frac{q}{M}\int_{-\infty}^{t}dt'\left[\ [\vec{E}+\frac{1}{w}\Big((\vec{v}'\cdot\vec{E})\vec{k}-(\vec{v}'\cdot\vec{k})\vec{E}\Big)]\cdot\nabla_v f_o(\vec{v}')\right]\ e^{i(\vec{k}\cdot\vec{r}'-wt')} \tag{98}$$

As before, a change of variable is made for t'- t. Using $t= t'$- t and the relationships for the position and velocities in section 2.1, equation (98) can be converted to a simple integral over τ. To do this, start by working with the term in the square bracket and completing the vector operations. Then equation (98), the term in the bracket, becomes

$$
\begin{aligned}
&[\vec{E}+\frac{1}{w}\left((\vec{v}'\cdot\vec{E})\vec{k}-(\vec{v}'\cdot\vec{k})\vec{E}\right)]\cdot\nabla_v f_o(\vec{v}') = \\
&\quad\left(\left[E_x+\frac{1}{w}\left(v_x'E_x+v_y'E_y+v_z'E_z\right)k_x-\frac{1}{w}\left(v_x'k_x+v_y'k_y+v_z'k_z\right)E_x\right]\ \hat{x}+\right. \\
&\quad\left[E_y+\frac{1}{w}\left(v_x'E_x+v_y'E_y+v_z'E_z\right)k_y-\frac{1}{w}\left(v_x'k_x+v_y'k_y+v_z'k_z\right)E_y\right]\ \hat{y}+ \\
&\quad\left.\left[E_z+\frac{1}{w}\left(v_x'E_x+v_y'E_y+v_z'E_z\right)k_z-\frac{1}{w}\left(v_x'k_x+v_y'k_y+v_z'k_z\right)E_z\right]\ \hat{z}\right)\cdot \\
&\quad\left(-\frac{v_x'}{v_{t\perp}^2}\hat{x}-\frac{v_t'}{v_{t\perp}^2}\hat{y}-\frac{(v_z'-v_o)}{v_{t\|}^2}\hat{z}\right)f_o(\vec{v}')
\end{aligned}
\tag{99}
$$

$$
\begin{aligned}
&[\vec{E}+\frac{1}{w}\left((\vec{v}'\cdot\vec{E})\vec{k}-(\vec{v}'\cdot\vec{k})\vec{E}\right)]\cdot\nabla_v f_o(\vec{v}') = \\
&\quad\frac{-1}{v_{t\perp}^2}\left[E_xv_x'+\frac{1}{w}\left(v_x'E_x+v_y'E_y+v_z'E_z\right)k_xv_x'+\frac{1}{w}\left(v_x'k_x+v_y'k_y+v_z'k_z\right)E_xv_x'\right]+ \\
&\quad\frac{-1}{v_{t\perp}^2}\left[E_yv_y'+\frac{1}{w}\left(v_x'E_x+v_y'E_y+v_z'E_z\right)k_yv_y'+\frac{1}{w}\left(v_x'k_x+v_y'k_y+v_z'k_z\right)E_yv_y'\right]+ \\
&\quad\frac{-1}{v_{t\|}^2}\left[\begin{array}{l}E_z(v_z'-v_o)+\frac{1}{w}\left(v_x'E_x+v_y'E_y+v_z'E_z\right)k_z(v_z'-v_o)+ \\ \frac{1}{w}\left(v_x'k_x+v_y'k_y+v_z'k_z\right)E_z(v_z'-v_o)\end{array}\right]
\end{aligned}
\tag{100}
$$

Working through the algebra and collecting terms for each electric field component yields

$$
\begin{aligned}
&[\vec{E}+\frac{1}{w}\left((\vec{v}'\cdot\vec{E})\vec{k}-(\vec{v}'\cdot\vec{k})\vec{E}\right)]\cdot\nabla_v f_o(\vec{v}') = \\
&\quad\frac{E_xv_x'}{v_{t\perp}^2}\left(1-\frac{v_z'k_z}{w}+\frac{v_{t\perp}^2(v_z'-v_o)k_z}{v_{t\|}^2w}\right)+ \\
&\quad\frac{E_yv_y'}{v_{t\perp}^2}\left(1-\frac{v_z'k_z}{w}+\frac{v_{t\perp}^2(v_z'-v_o)k_z}{v_{t\|}^2w}\right)+ \\
&\quad E_z\left[\frac{(v_z'-v_o)}{v_{t\|}^2}\left(1-\frac{k_yv_y'}{w}+\frac{k_xv_x'}{w}\right)-\frac{k_yv_y'v_z'}{v_{t\perp}^2w}+\frac{k_xv_x'v_z'}{v_{t\perp}^2w}\right]
\end{aligned}
\tag{101}
$$

For simplicity, both the electric field and the wave vector are converted to cylindrical coordinates. In doing so, the result applies only to electromagnetic waves that are cylindrically symmetric, i. e., circularly polarized waves. For linearly polarized waves, this transformation does not work and the calculation would have to be carried out more generally. Using

$$\begin{aligned} E_x &= E_\perp \cos f \\ E_y &= E_\perp \sin f \\ k_x &= k_\perp \cos f \\ k_y &= k_\perp \sin f \end{aligned} \tag{102}$$

and $k_z=k_{||}$, equation (101) can be written as

$$\begin{aligned} &[\vec{E}+\frac{1}{w}\left((\vec{v}'\cdot\vec{E})\vec{k}-(\vec{v}'\cdot\vec{k})\vec{E}\right)]\cdot\nabla_v f_o(\vec{v}')= \\ &\quad\left(\frac{E_\perp \cos f v_x'}{v_{t\perp}^2}+\frac{E_\perp \sin f v_y'}{v_{t\perp}^2}\right)\left(1-\frac{v_z' k_{||}}{w}+\frac{v_{t\perp}^2(v_z'-v_o)k_{||}}{v_{t||}^2 w}\right) \\ &\quad+E_z\left[\frac{(v_z'-v_o)}{v_{t||}^2}\left(1-\frac{k_\perp \sin f v_y'}{w}+\frac{k_\perp \cos f v_x'}{w}\right)-\frac{k_\perp \sin f v_y' v_z'}{v_{t\perp}^2 w}+\frac{k_\perp \cos f v_x' v_z'}{v_{t\perp}^2 w}\right] \end{aligned} \tag{103}$$

$$\begin{aligned} &[\vec{E}+\frac{1}{w}\left((\vec{v}'\cdot\vec{E})\vec{k}-(\vec{v}'\cdot\vec{k})\vec{E}\right)]\cdot\nabla_v f_o(\vec{v}')= \\ &\quad\frac{E_\perp}{v_{t\perp}^2}\left(\cos f v_x'+\sin f v_y'\right)\left(1-\frac{v_z' k_{||}}{w}+\frac{v_{t\perp}^2(v_z'-v_o)k_{||}}{v_{t||}^2 w}\right) \\ &\quad+E_z\left[\frac{(v_z'-v_o)}{v_{t||}^2}\left(1-\frac{k_\perp}{w}\left(\sin f v_y'+\cos f v_x'\right)\right)-\frac{k_\perp v_z'}{v_{t\perp}^2 w}\left(\sin f v_y'+\cos f v_x'\right)\right] \end{aligned} \tag{104}$$

Only terms containing v_x' and v_y' are of the form

$$\sin f v_y' + \cos f v' \tag{105}$$

Using the relationship for v_x' and v_y' from section 2.0 this can be written as

$$\sin f v_y' + \cos f v' = \sin f\left(v_x \cos(\Omega t) - v_y \sin(\Omega t)\right) + \cos f\left(v_x \sin(\Omega t) + v_y \cos(\Omega t)\right) \tag{106}$$

With the help of some trigonometry equation (106) becomes

$$\sin f v_y' + \cos f v' = v_x \cos\left(f - e\Omega t\right) + v_y \cos\left(f - e\Omega t + p/2\right) \tag{107}$$

Using equation (107), $v_z' = v_z$, and $v_{t\wedge}^2/v_{t||}^2 = T_\wedge/T_{||}$,

$$[\vec{E} + \frac{1}{w}\left((\vec{v}'\cdot\vec{E})\vec{k} - (\vec{v}'\cdot\vec{k})\vec{E}\right)]\cdot\nabla_v f_o(\vec{v}') = \left(A\left(v_x\cos(f - e\Omega t) + v_y\cos(f - e\Omega t + p/2)\right) + B\right)\frac{f_o(\vec{v})}{v_{t\perp}^2} \tag{108}$$

where

$$A = E_\perp\left(1 - \frac{v_z k_\|}{w} + \frac{T_\perp}{T_\|}\left(\frac{(v_z - v_o)k_\|}{w}\right)\right) + E_z k_\perp\left(\frac{v_z}{w} - \frac{T_\perp}{T_\|}\left(\frac{(v_z - v_o)}{w}\right)\right) \tag{109}$$

and

$$B = E_z\frac{T_\perp}{T_\|}(v_z - v_o) \tag{110}$$

Substituting equation (109) and equation (18) into equation (98) and changing the integration variable yields

$$\begin{aligned} f_1(\vec{v},t) = -\frac{q}{M}\int_{-\infty}^{0} dt\left(A\left(v_x\cos(f - e\Omega t) + v_y\cos(f - e\Omega t + p/2)\right) + B\right)\frac{f_o(\vec{v})}{v_{t\perp}^2} \\ \times e^{i(\vec{k}\cdot\vec{r} - wt)} e^{i\frac{ek_\perp}{\Omega}(v_y\cos f - v_x\sin f)} \sum_{m,n=-\infty}^{\infty} J_m\left(\frac{k_\perp v_y}{\Omega}\right)J_n\left(\frac{k_\perp v_x}{\Omega}\right) \\ \times e^{i(n+m)f + imp/2} e^{i(w - e(n+m)\Omega - k_\| v_z)t} \end{aligned} \tag{111}$$

Rearranging this equation for convenience, it becomes

$$\begin{aligned} f_1(\vec{v},t) = -e^{i(\vec{k}\cdot\vec{r} - wt)} e^{i\frac{ek_\perp}{\Omega}(v_y\cos f - v_x\sin f)} \frac{q}{M}\frac{f_o(\vec{v})}{v_{t\perp}^2}\int_{-\infty}^{0} dt \sum_{m,n=-\infty}^{\infty} J_m\left(\frac{k_\perp v_y}{\Omega}\right)J_n\left(\frac{k_\perp v_x}{\Omega}\right) \\ \times\left(A\left(v_x\cos(f - e\Omega t) + v_y\cos(f - e\Omega t + p/2)\right) + B\right) \\ \times e^{i(n+m)f + imp/2} e^{i(w - e(n+m)\Omega - k_\| v_z)t} \end{aligned} \tag{112}$$

The next three steps help reduce equation (112) into a simple integral over τ. These steps expand the cosine terms into exponential form and expand the Bessel functions.

$$\begin{aligned} f_1(\vec{r},\vec{v},t) = -e^{i(\vec{k}\cdot\vec{r} - wt)} e^{i\frac{ek_\perp}{\Omega}(v_y\cos f - v_x\sin f)} \frac{q}{M}\frac{f_o(\vec{v})}{v_{t\perp}^2}\int_{-\infty}^{0} dt \sum_{m,n=-\infty}^{\infty} J_m\left(\frac{k_\perp v_y}{\Omega}\right)J_n\left(\frac{k_\perp v_x}{\Omega}\right) \\ \times\left[\frac{Av_x}{2}\left(e^{i(f - e\Omega t)} + e^{-i(f - e\Omega t)}\right) + \frac{Av_y}{2}\left(e^{i(f - e\Omega t + p/2)} + e^{-i(f - e\Omega t + p/2)}\right) + \right. \\ \left. B\right] e^{i(w + e(n+m)\Omega - k_z v_z)t - i(n+m)f + imp/2} \end{aligned} \tag{113}$$

$$f_1(\vec{v},t) = -e^{i(\vec{k}\cdot\vec{r}-wt)} e^{i\frac{ek_\perp}{\Omega}(v_y\cos f - v_x \sin f)} \frac{q}{M}\frac{f_o(\vec{v})}{v_{t\perp}^2}\int_{-\infty}^{0} dt \sum_{m,n=-\infty}^{\infty} J_m(\frac{k_\perp v_y}{\Omega})J_n(\frac{k_\perp v_x}{\Omega})$$

$$\times\Bigg(\frac{Av_x}{2}e^{im\pi/2}\left(e^{i(w+e(n+1+m)\Omega-k_\parallel v_z)t-i(n+1+m)f}+e^{-i(w+e(n-1+m)\Omega-k_\parallel v_z)t-i(n-1+m)f}\right)$$
$$+\frac{Av_y}{2}\left(e^{i(w+e(n+m+1)\Omega-k_\parallel v_z)t-i(n+m+1)f+i(m+1)\pi/2}+e^{-i(w+e(n+m-1)\Omega-k_\parallel v_z)t-i(n+m-1)f+i(m-1)\pi/2}\right)$$
$$+Be^{i(w+e(n+m)\Omega-k_\parallel v_z)t-i(n+m)f+im\pi/2}\Bigg) \quad (114)$$

$$f_1(\vec{v},t) = -e^{i(\vec{k}\cdot\vec{r}-wt)} e^{i\frac{ek_\perp}{\Omega}(v_y\cos f - v_x \sin f)} \frac{q}{M}\frac{f_o(\vec{v})}{v_{t\perp}^2}\int_{-\infty}^{0} dt \sum_{m,n=-\infty}^{\infty}$$
$$\frac{Av_x}{2}J_m(\frac{k_\perp v_y}{\Omega})\left(J_{n+1}(\frac{k_\perp v_x}{\Omega})+J_{n-1}(\frac{k_\perp v_x}{\Omega})\right)e^{i(w+e(n+m)\Omega-k_\parallel v_z)t-i(n+m)f+im\pi/2}$$
$$+\frac{Av_y}{2}J_n(\frac{k_\perp v_x}{\Omega})\left(J_{m+1}(\frac{k_\perp v_y}{\Omega})+J_{m-1}(\frac{k_\perp v_y}{\Omega})\right)e^{i(w+e(n+m)\Omega-k_\parallel v_z)t-i(n+m)f+im\pi/2}$$
$$+BJ_m(\frac{k_\perp v_y}{\Omega})J_n(\frac{k_\perp v_x}{\Omega})e^{i(w+e(n+m)\Omega-k_\parallel v_z)t-i(n+m)f+im\pi/2} \quad (115)$$

Using the Bessel identity from equation (31), equation (115) becomes

$$f_1(\vec{v},t) = -e^{i(\vec{k}\cdot\vec{r}-wt)} e^{i\frac{ek_\perp}{\Omega}(v_y\cos f - v_x \sin f)} \frac{q}{M}\frac{f_o(\vec{v})}{v_{t\perp}^2}\int_{-\infty}^{0} dt \sum_{m,n=-\infty}^{\infty}$$
$$\left(\frac{Av_x}{2}J_m(\frac{k_\perp v_y}{\Omega})\left(\frac{2n\Omega}{k_\perp v_x}J_n(\frac{k_\perp v_x}{\Omega})\right)+\frac{Av_y}{2}J_n(\frac{k_\perp v_x}{\Omega})\left(\frac{2m\Omega}{k_\perp v_y}J_m(\frac{k_\perp v_y}{\Omega})\right)+\right. \quad (116)$$
$$\left.BJ_m(\frac{k_\perp v_y}{\Omega})J_n(\frac{k_\perp v_x}{\Omega})\right)e^{i(w+e(n+m)\Omega-k_\parallel v_z)t-i(n+m)f+im\pi/2}$$

$$f_1(\vec{v},t) = -e^{i(\vec{k}\cdot\vec{r}-wt)} e^{i\frac{ek_\perp}{\Omega}(v_y\cos f - v_x \sin f)} \frac{q}{M}\frac{f_o(\vec{v})}{v_{t\perp}^2}\sum_{m,n=-\infty}^{\infty} J_m(\frac{k_\perp v_y}{\Omega})J_n(\frac{k_\perp v_x}{\Omega})$$
$$\left(A\left(\frac{(n+m)\Omega}{k_\perp}\right)+B\right)e^{-i(n+m)f+im\pi/2}\int_{-\infty}^{0} dte^{i(w+e(n+m)\Omega-k_\parallel v_z)t} \quad (117)$$

Now doing the integral over τ equation (117) becomes

$$f_1(\vec{v},t) = -e^{i(\vec{k}\cdot\vec{r}-wt)} e^{i\frac{ek_\perp}{\Omega}(v_y\cos f - v_x\sin f)} \frac{q}{M}\frac{f_o(\vec{v})}{v_{t\perp}^2} \sum_{m,n=-\infty}^{\infty} J_m\left(\frac{k_\perp v_y}{\Omega}\right) J_n\left(\frac{k_\perp v_x}{\Omega}\right)$$
$$\times e^{-i(n+m)f + imp/2} \left(A\left(\frac{(n+m)\Omega}{k_\perp}\right) + B \right) \frac{-i}{w + e(n+m)\Omega - k_\| v_z} \qquad (118)$$

Substituting back in for A and B equation (118), it becomes

$$f_1(\vec{v},t) = -e^{i(\vec{k}\cdot\vec{r}-wt)} \frac{q}{M}\frac{f_o(\vec{v})}{v_{t\perp}^2} \sum_{m,n=-\infty}^{\infty} J_m\left(\frac{k_\perp v_y}{\Omega}\right) J_n\left(\frac{k_\perp v_x}{\Omega}\right) e^{imp/2} e^{i(n+m)f}$$
$$\left(\left(E_\perp \left(1 - \frac{v_z k_\|}{w} + \frac{T_\perp}{T_\|}\left(\frac{(v_z - v_o)k_\|}{w}\right)\right) + E_z k_\perp \left(\frac{v_z}{w} - \frac{T_\perp}{T_\|}\left(\frac{(v_z - v_o)}{w}\right)\right)\right)\left(\frac{(n+m)\Omega}{k_\perp}\right) + \right. \qquad (119)$$
$$\left. + E_z \frac{T_\perp}{T_\|}(v_z - v_o) \right) \frac{i}{e(n+m)\Omega + (k_\| v_z - w)}$$

Rearranging this equation for simplicity gives

$$f_1(\vec{v},t) = -e^{i(\vec{k}\cdot\vec{r}-wt)} e^{i\frac{ek_\perp}{\Omega}(v_y\cos f - v_x\sin f)} \frac{q}{M}\frac{f_o(\vec{v})}{v_{t\perp}^2} \sum_{m,n=-\infty}^{\infty} J_m\left(\frac{k_\perp v_y}{\Omega}\right) J_n\left(\frac{k_\perp v_x}{\Omega}\right)$$
$$\times e^{imp/2} e^{i(n+m)f} \left[\left(E_\perp + \left(E_\perp k_\| - E_z k_\perp\right)\left(\frac{T_\perp}{T_\|}\left(\frac{(v_z - v_o)}{w}\right) - \frac{v_z}{w}\right)\right)\left(\frac{(n+m)\Omega}{k_\perp}\right) \right. \qquad (120)$$
$$\left. + E_z \frac{T_\perp}{T_\|}(v_z - v_o) \right] \frac{i}{e(n+m)\Omega + (k_\| v_z - w)}$$

Equation (120) is the first order perturbed velocity distribution function for a general circularly polarized electromagnetic wave. Looking at this equation, differences with the electrostatic $f_1(v)$ can be seen. Equation (120) has two parts, an electrostatic part and an electromagnetic part. The electromagnetic part is the curl term, $k_\wedge E_z$-$k_{||}E_\wedge$, and the electrostatic terms are the individual $E_\wedge$ and E_z terms. If we let the curl term go to zero, $E_\wedge = k_\wedge E_z/k_z$, equation (120) reduces to equation (35), $f_1(v)$ for an electrostatic wave. Thus, integration over the velocities, $f_1(v)$ can yield the perturbed distribution as a function of one component of velocity.

3.1 Integrating f_1(r,v,t) over velocity components

Equation (120) can be separated as equation (35) to collect the terms with each component of velocity together. The v_x and v_y terms are the same as in equation (35), so the results for the integration of these terms from the electrostatic case can be used for the electromagnetic case. The only velocity component that is different from the electrostatic case is v_z as can be seen in equation (121). In this section the integration over v_z is done.

3.1.1 Integration of v_z

The integral over v_z is

$$
\begin{aligned}
f_1(\vec{r},\vec{v},t) &= \left(\frac{-ie}{Mv_{t\perp}^2}\right) e^{-i(\vec{k}\cdot\vec{r}-wt)} \sum_{m,n} e^{-imp/2} e^{i(m+n)f} \\
&\times\left[\left(\frac{1}{2p\, v_{t\perp}^2}\right)^{1/2} e^{-v_y^2/2v_{t\perp}} e^{iek_x v_y/\Omega} J_m\left(\frac{k_\perp v_y}{\Omega}\right)\right] \\
&\times\left[\left(\frac{1}{2p\, v_{t\perp}^2}\right)^{1/2} e^{-v_x^2/2v_{t\perp}} e^{-iek_y v_x/\Omega} J_n\left(\frac{k_\perp v_x}{\Omega}\right)\right]
\end{aligned}
\tag{121}
$$

$$
\begin{aligned}
&\times\left\{\left(\frac{1}{2p\, v_{t\|}^2}\right)^{1/2} e^{-(v_z-v_o)^2/2v_{t\|}} \left[\left(E_\perp + (k_\| E_\perp + k_\perp E_\|)\left(\frac{T_\perp}{T_\|}\frac{(v_z-v_o)}{w} - \frac{v_z}{w}\right)\right)\right.\right. \\
&\qquad \left.\left.\times\frac{(n+m)\Omega}{k_\perp} + E_z\left(\frac{T_\perp}{T_\|}(v_z-v_o)\right)\right]\left(\frac{1}{e(n+m)\Omega + k_\| v_z - w}\right)\right\}
\end{aligned}
$$

Using the last term in equation (121), the integral over v_z is

$$
\begin{aligned}
&\left(\frac{1}{2p v_{t\|}^2}\right)^{1/2} \int_{-\infty}^{\infty} dv_z e^{-(v_z-v_o)^2/2v_{t\|}} \left[\left(E_\perp + (k_\| E_\perp + k_\perp E_\|)\left(\frac{T_\perp}{T_\|}\frac{(v_z-v_o)}{w} - \frac{v_z}{w}\right)\right)\right. \\
&\qquad \left.\times\frac{(n+m)\Omega}{k_\perp} + E_z\left(\frac{T_\perp}{T_\|}(v_z-v_o)\right)\right]\left(\frac{1}{e(n+m)\Omega + k_\| v_z - w}\right)\Bigg\}
\end{aligned}
\tag{122}
$$

This integral is completed by first making the following substitution

$$
\begin{aligned}
s &= \frac{v_z - v_o}{\sqrt{2}\ \ v_{t\|}} \\
dv_z &= \sqrt{2}\ \ v_{t\|} ds
\end{aligned}
\tag{123}
$$

With this substitution equation (122) becomes

$$\left(\frac{2v_{t||}^2}{p}\right)^{1/2}\int_{-\infty}^{\infty} ds e^{-s^2}\left[\left(\frac{E_\perp}{\sqrt{2}v_{t||}}+(\frac{k_{||}E_\perp+k_\perp E_{||}}{w})\left(\frac{T_\perp}{T_{||}}s-s+\frac{v_o}{\sqrt{2}v_{t||}}\right)\right)\frac{(n+m)\Omega}{k_\perp}+E_z\left(\frac{T_\perp}{T_{||}}s\right)\right] \times\left(\frac{1}{k_{||}(s+d_{m+n})}\right) \tag{124}$$

where d_{m+n} is as before

$$d_{n+m}=\frac{w-e(n+m)\Omega-k_{||}v_o}{\sqrt{2}\,k_{||}v_{t||}} \tag{125}$$

The integral in equation (124) can be completed using the plasma dispersion relationships in equations (42) and (43). The result of the integration is

$$\frac{\sqrt{2}v_{t||}}{k_{||}}\left[\left(\frac{E_\perp}{\sqrt{2}v_{t||}}Z(d_{m+n})+(\frac{k_{||}E_\perp+k_\perp E_{||}}{w})\left(\left(-\frac{1}{2}\right)\left(\frac{T_\perp}{T_{||}}-1\right)Z'(d_{m+n})+\frac{v_o}{\sqrt{2}v_{t||}}Z(d_{m+n})\right)\right) \times\frac{(n+m)\Omega}{k_\perp}+E_z\frac{T_\perp}{T_{||}}\left(-\frac{1}{2}\right)Z'(d_{m+n})\right] \tag{126}$$

Using the following plasma dispersion function relationship

$$Z'(d)=-2[1+dZ(d)] \tag{127}$$

equation (126) can be rewritten as

$$\frac{\sqrt{2}v_{t||}}{k_{||}}\left[\left(\frac{E_\perp}{\sqrt{2}v_{t||}}Z(d_{m+n})+(\frac{k_{||}E_\perp+k_\perp E_{||}}{w})\left(\left(\frac{T_\perp}{T_{||}}-1\right)(1+d_{m+n}Z(d_{m+n}))+\frac{v_o}{\sqrt{2}v_{t||}}Z(d_{m+n})\right)\right) \times\frac{(n+m)\Omega}{k_\perp}+E_z\frac{T_\perp}{T_{||}}(1+d_{m+n}Z(d_{m+n}))\right] \tag{128}$$

Thus, the integral over v_z is

$$\left(\frac{1}{2pv_{t||}^2}\right)^{1/2}\int_{-\infty}^{\infty} dv_z e^{-(v_z-v_o)^2/2v_{t||}}\left[\left(E_\perp+(k_{||}E_\perp+k_\perp E_{||})\left(\frac{T_\perp}{T_{||}}\frac{(v_z-v_o)}{w}-\frac{v_z}{w}\right)\right) \times\frac{(n+m)\Omega}{k_\perp}+E_z\left(\frac{T_\perp}{T_{||}}(v_z-v_o)\right)\right]\left(\frac{1}{e(n+m)\Omega+k_{||}v_z-w}\right\}= \tag{129}$$

$$\frac{\sqrt{2}v_{t||}}{k_{||}}\left[\left(\frac{E_\perp}{\sqrt{2}v_{t||}}Z(d_{m+n})+(\frac{k_{||}E_\perp+k_\perp E_{||}}{w})\left(1-\frac{T_\perp}{T_{||}}-\left[\left(1-\frac{T_\perp}{T_{||}}\right)d_{m+n}-\frac{v_o}{\sqrt{2}v_{t||}}\right]Z(d_{m+n})\right)\right) \times\frac{(n+m)\Omega}{k_\perp}+E_z\frac{T_\perp}{T_{||}}(1+d_{m+n}Z(d_{m+n}))\right]$$

3.1.2 Single Velocity Component Electromagnetic $f_1(r,v,t)$

Using the integral over v_z in this section, equation (68), and the integrals over v_x and v_y from sections 2.1.2 and 2.1.3, a perturbed velocity distribution function for an electromagnetic wave can be written as a function of only v_y (or v_x, see discussion in section 2.1.4) and v_z. These equations are

$$
\begin{aligned}
f_1(\vec{r},v_y,t) = \left(\frac{-e}{p\,M\,v_{t\perp}^{\,2}}\right) e^{-i(\vec{k}\cdot\vec{r}-wt)} f_o(v_y) \sum_{n,m} \frac{\sqrt{2}v_{t\|}}{k_\|} \Bigg[\Bigg(& \frac{E_\perp}{\sqrt{2}v_{t\|}} Z(d_{m+n}) + \Big(\frac{k_\| E_\perp + k_\perp E_\|}{w}\Big) \\
& \times \left(1 - \frac{T_\perp}{T_\|} - \left[\left(1 - \frac{T_\perp}{T_\|} \right) d_{m+n} - \frac{v_o}{\sqrt{2}v_{t\|}} \right] Z(d_{m+n}) \right) \Bigg) \\
& \times \frac{(n+m)\Omega}{k_\perp} + E_z \frac{T_\perp}{T_\|} (1 + d_{m+n} Z(d_{m+n})) \Bigg] \\
& \times J_m\Big(\frac{k_\perp v_y}{\Omega}\Big) e^{-imp/2} e^{i(m+n)q}\, e^{iek_x v_y/\Omega} e^{-a^2/8} e^{-c^2/4} \\
& \times \sum_p I_{(n-p)/2}\Big(\frac{a^2}{8}\Big) I_p\Big(\frac{e\,ac}{2}\Big) e^{-ipp/2}
\end{aligned}
\qquad (130)
$$

$$
\begin{aligned}
f_1(\vec{r},v_z,t) = \Big(\frac{-ie}{Mv_{t\perp}^2}\Big) & e^{-i(\vec{k}\cdot\vec{r}-wt)} f_o(v_z) \sum_{m,n} e^{-imp/2} e^{i(m+n)f} \\
& \times e^{-d^2/4} \sum_{p=-\infty}^{\infty} I_p\left(\frac{ead}{2}\right) I_{\frac{m-p}{2}}\left(\frac{a^2}{8}\right) e^{-ip\frac{p}{2}} \\
& \times e^{-c^2/4} \sum_{l=-\infty}^{\infty} I_l\left(\frac{e\,ac}{2}\right) I_{\frac{n-l}{2}}\left(\frac{a^2}{8}\right) e^{-il\frac{p}{2}} \\
& \times \Bigg\{ \Bigg[\left(E_\perp + (k_\| E_\perp + k_\perp E_\|) \left(\frac{T_\perp}{T_\|} \frac{(v_z - v_o)}{w} - \frac{v_z}{w} \right) \right) \\
& \quad \times \frac{(n+m)\Omega}{k_\perp} + E_z \left(\frac{T_\perp}{T_\|} (v_z - v_o) \right) \Bigg] \left(\frac{1}{e(n+m)\Omega + k_\| v_z - w} \right) \Bigg\}
\end{aligned}
\qquad (131)
$$

where all symbols have been defined in previous sections.

4 Laser Induced Florescence measurement of $f_1(r,v,t)$

Now that the theoretical calculation of the perturbed distribution has been completed, the technique for measuring the perturbed distribution is described. This section will discuss the measurement of $f_1(\mathbf{r},v,t)$. There are three possible techniques that can be used to measure $f_1(\mathbf{r},v,t)$. The most common and the simplest method uses a lock-in amplifier. This, along with the basics for measuring $f_0(v)$ will be discussed in section 4.1. Section 4.2 will discuss using a digitizer to measure $f_1(\mathbf{r},v,t)$ and the last section, 4.3, will discuss the possibilities of using a cross power spectrum technique for the measurement.

4.1 Measurement with a Lock-in amplifier

Before discussing measurements of $f_1(\mathbf{r},v,t)$, a brief discussion of $f_o(\mathbf{r},v,t)$ measurements is needed. Typical LIF measurements in argon[3,9] pump the metastable state at $\lambda \approx 611.5$ nm, and collect the emitted light at $\lambda \approx 461.0$ nm as shown in the level diagram of Figure 3.

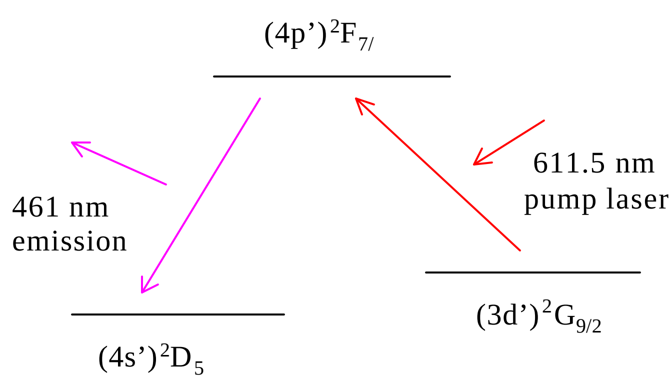

Figure 3: Commonly used LIF argon schematic.

By scanning the laser over a narrow frequency band that includes the $\lambda \approx 611.5$ nm absorption line while measuring the intensity of the emitted light, the line shape can be measured. The shape of this line is determined by several line broadening mechanisms,[9] but for ion temperatures greater than 0.02 eV and magnetic fields less the 1.2 kG Doppler broadening dominates. This allows a direct correlation between the ion velocity distribution function and shape of the collected light from the laser scan. The problem of background light at $\lambda \approx 461.0$ nm is defeated by chopping the laser light and using a lock-in amplifier to measure pumped emission at the chopping frequency. By only measuring the amplitude of the emission at the chopping frequency, the lock-in amplifier increases the signal to noise ratio. A typical schematic of the apparatus is shown in Figure 4.

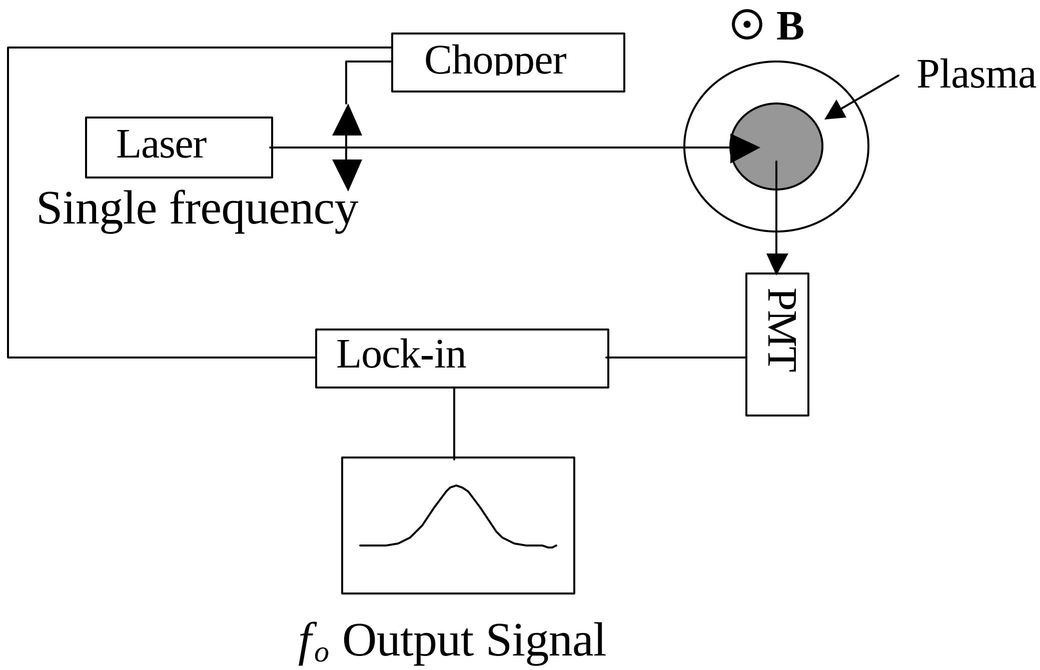

Figure 4: Typeical LIF schematic for measuring the zeroth order distribution function.

A typical LIF measurement made in the Hot hELIcon experiment (HELIX)[10,11] is shown in Figure 5. Here the x axis is the laser frequency and the width of the line is due to Doppler broadening. A fit of the light intensity curve using equation (132) can be used to determine the temperature of the ions along the axis of the laser's direction.

$$I(u) = I_o(u)\exp(-0.0779(u - u_o)^2 / T_{\mathrm{argon}}) \tag{132}$$

Here I_o is the peak intensity, n is laser frequency, n_o is laser center frequency, T_{argon} is the ion temperature, and the coefficient in the exponential, 0.0779, has been calculated specifically for argon.

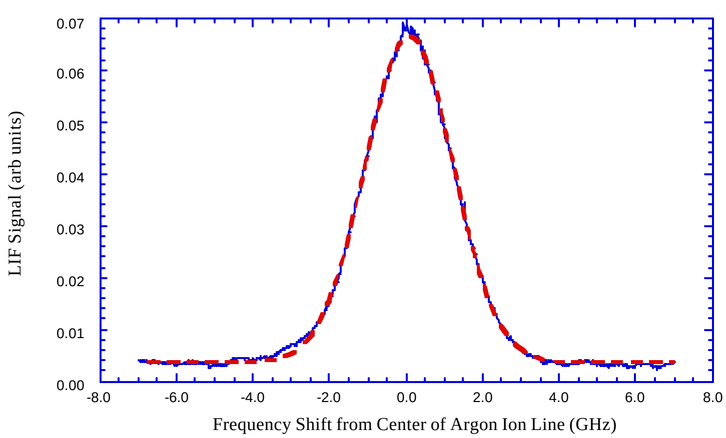

Figure 5: A typical LIF trace in Helix where the blue line is the signal and the red dots are the curve fit. This ion velocity distribution has a temperature of 0.21 eV .[12]

What is different about the measurement of first order perturbation to the velocity distribution function? The measurement of the zeroth order velocity distribution function only cares about the intensity of light at the chopped frequency as a function of velocity. The measurement of the first order perturbed velocity distribution function is concerned with the intensity of light fluctuating at the frequency of the perturbing wave. As a particle follows the path of its gyro-motion, its velocity is being affected by the wave. The wave is speeding it up and slowing it down at the frequency of the wave. These oscillations in the particles velocities change the number of particles at each particular velocity in a periodic manner, i. e., perturbations to the velocity distribution function. These perturbations are directly proportional to the fluctuation in the intensity of light for any fixed velocity or laser offset position. The intensity of the fluctuations or perturbation at a particular velocity tells how strongly the particles with that velocity couple to the wave. Scanning the laser over the velocity distribution function, the intensity of the perturbation as a function of velocity is measured. The intensity along with the relative phase of the oscillation or perturbation is a measurement of the first order perturbed velocity distribution function. A schematic of the experimental setup is shown in Figure 6. Notice that a chopper is not used for this measurement, but the lock-in reference signal is the signal going to the antenna used to launch the wave. This allows the lock-in to measure the perturbed distribution at the frequency of the wave driven in the plasma by the antenna. For the lock-in to function properly this the reference signal must be a clean sinusoidal or square wave.

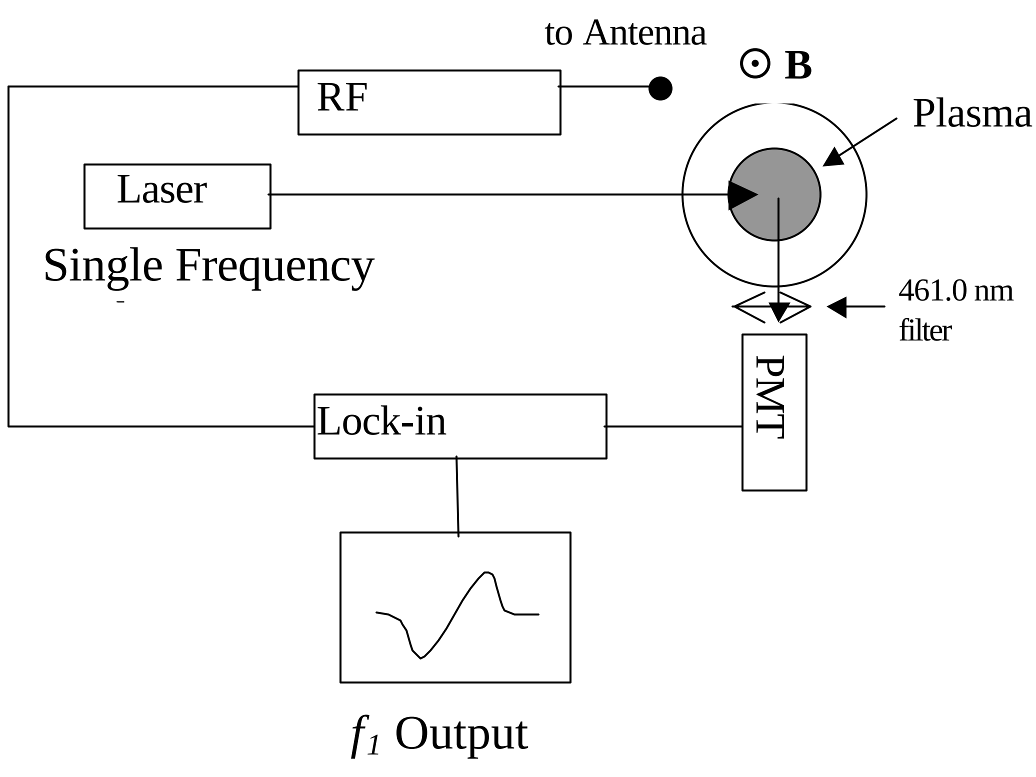

Figure 6: A schematic diagram for the measurement of the perturbed velocity distribution function.

Using this method, $f_1(\mathbf{r}, v_y, t)$ for ion cyclotron waves generated in HELIX by an additional heating antenna have been measured[13]. The measured $f_1(\mathbf{r}, v_y, t)$ along with the theoretical curves (using the theory from sect. 2) are shown in Figure 7. The fit gives wave numbers of $k_\wedge \sim 0.150$ cm^{-1} and $k_{||} \sim 0.404$ cm^{-1}. These wave numbers are consistent with electrostatic ion cyclotron wave in a plasma with a 4 eV electron temperature as in HELIX.

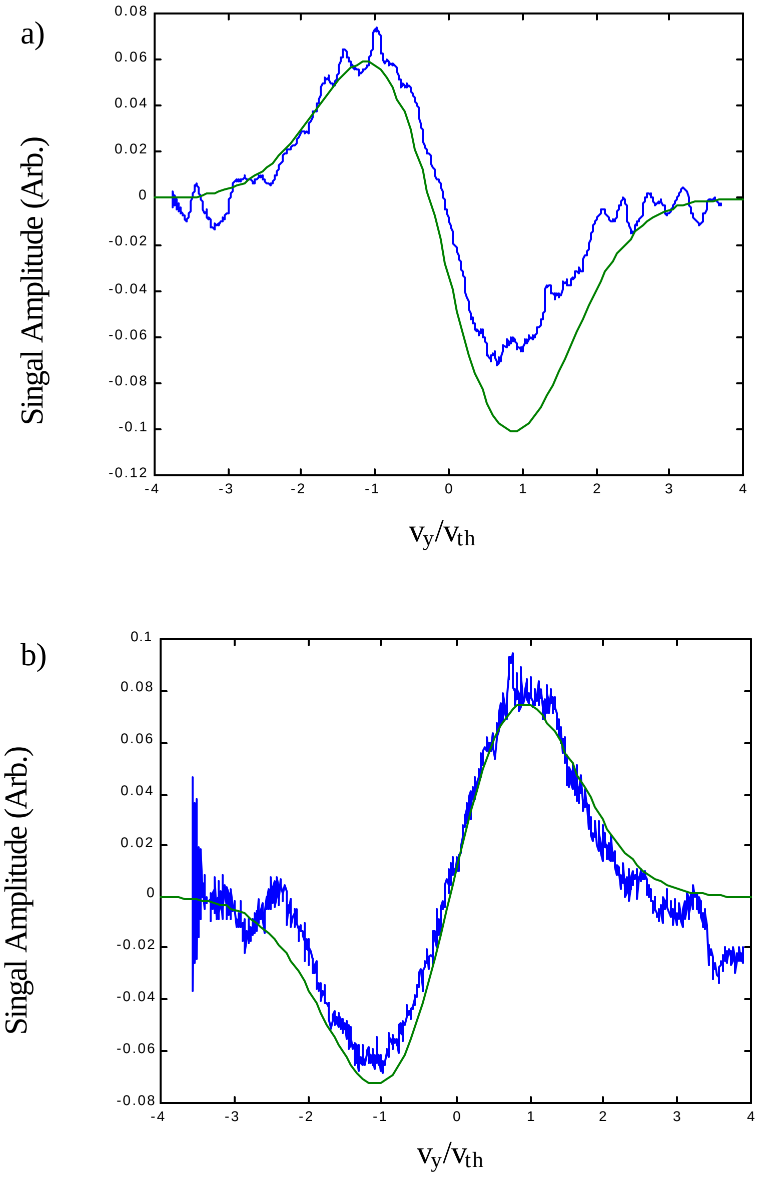

Figure 7: a) shows the measured and theory curves (smooth lines) for the real part of the $f_1(\mathbf{r}, v_y, t)$. b) shows the measured and theory curves for the imaginary part of the $f_1(\mathbf{r}, v_y, t)$.

4.2 Measurment Using Averaged Time Series

An alternate method to using a lock-in amplifier is to use a digitizer to record the PMT signal directly. This method not only provides information about $f_1(\mathbf{r},\mathbf{v},\mathbf{t})$, but can produce information about $f_o(v)$and perturbed distributions at other frequencies besides the reference frequency. This information can not be obtained directly with a single lock-in based measurement. A lock-in amplifier uses the reference signal to generate a sine and cosine signal at a single reference frequency. The lock-in takes each of these signals, multiplies them with the input signal, and integrates for a fixed length of time (integration time on lock-in). The result is the Fourier amplitude of the input signal at the reference frequency. The sine and cosine parts give the real and imaginary parts of the Fourier amplitudes, which can easily be turned into amplitude and phase. With a lock-in, this is done using precision analog electronics. In this day and age of digital electronics, it is possible to do this electronically. However, the precision electronics in the lock-in amplifier can pick out small signals, i. e., yield good signal-to-noise ratios for small signals. With the digitizer, this is done by averaging multiple time series. Since perturbed distribution measurements are of signals that are phase coherent with a driving wave (signals that keep the same phase for a given velocity over time), the digitizer can be triggered by a reference signal (from the driving antenna) at the same point in time while directly recording the PMT signal for several time series. These time series are averaged to bring the signal out of the noise, since only signals that have the same phase in each time series will add while random signals will cancel out. This technique is similar to boxcar averaging or using a multi-channel averaging. For perturbed velocity distribution measurements, an averaged time series is taken for several different velocities or laser offset frequencies resolving the velocity dependence of the distributions. An example of this measurement is shown below in Figure 8. The colors in the plot represent the intensity of the time averaged PMT signal as a function of time and velocity or laser frequency.

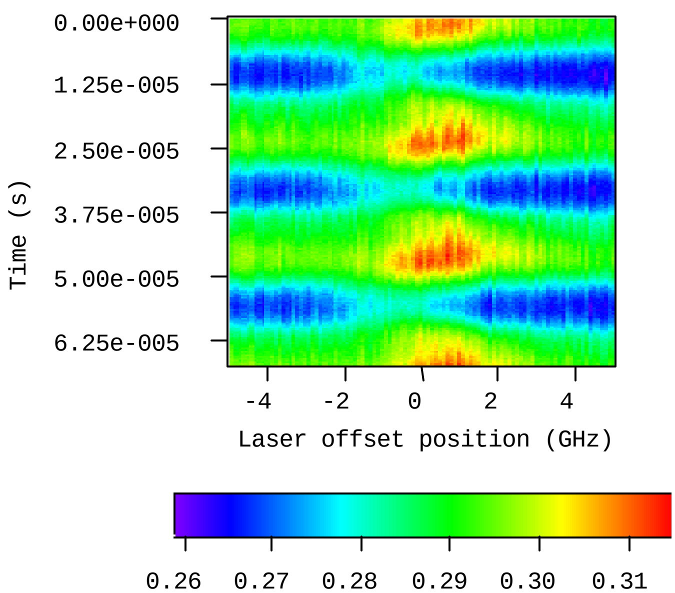

Figure 8: Intensity plot of the averaged time series as a function of velocity or laser frequency.

Figure 8 clearly shows the PMT signal is periodic. This periodicity corresponds the driving frequency and it is clear that approximately three wave periods are recorded along the time axis. Looking across the laser offset frequency axis, it is clear that the signal intensity has some velocity dependence. This is evidence that there is a coherent LIF signal in this data at least at the driving frequency. Note that on the laser offset frequency axis, there is a periodic PMT signal for the laser greater than 4 GHz and less than –4 GHz. The laser offset frequency is outside of the Doppler broadened region where there is no LIF signal, but there is spontaneous emission at 461.0 nm from the ions in the plasma. The motion of these ions is affected by the wave making the emitted light periodic in nature. This effect is also seen with the lock-in amplifier because the Lock-in signal does not go to zero outside of the Doppler broadened region. The signal from the background light is subtracted out in the analysis.

To analyze the data, the data are converted from the time domain to the frequency domain since the perturbed velocity distributions are a function of frequency. To put the data from Figure 8 into frequency space, the FFT of each averaged time series is taken and plotted against velocity giving both a plot of the real and the imaginary parts of the FFT. The perturbed velocity distribution is the signal amplitude as a function of velocity for a single frequency. If there are any signals that are phase coherent with the reference (digitizer triggering signal) at any frequency, a clear signal can be seen in a plot of the signal amplitude versus velocity. This is the advantage over the lock-in method which can only measure $f_1(\mathbf{r},v,t)$ at the reference frequency. To give a better illustration of where the perturbed velocity distributions are in the data from Figure 8, the power spectrum of each averaged time series is done and the signal amplitude is plotted versus frequency and velocity in Figure 9.

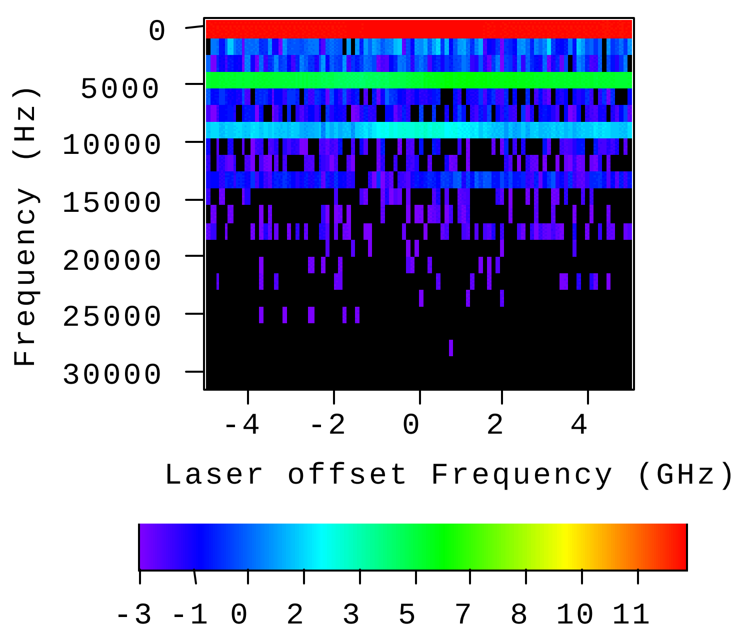

Figure 9: Intensity plot of the power spectrum as function of frequency and velocity or laser frequency.

From the intensity plot in Figure 9, there are four clear bands that occur at unique frequencies and for all velocities. These are the frequencies where clear phase coherent information is, or the frequencies with the largest Fourier components. It is also clear from the equal spacing of these bands along the frequency axis that the signals are at harmonic frequencies of the driving frequency. These measurements were made using 35.5 kHz driving signal to launch the waves. Figure 10 has graphs of the real and imaginary parts of the FFTs generated from the data shown in Figure 8 for the frequencies with the most signal in Figure 9. In each of the graphs in Figure 10, the background has been subtracted out from the FFT information.

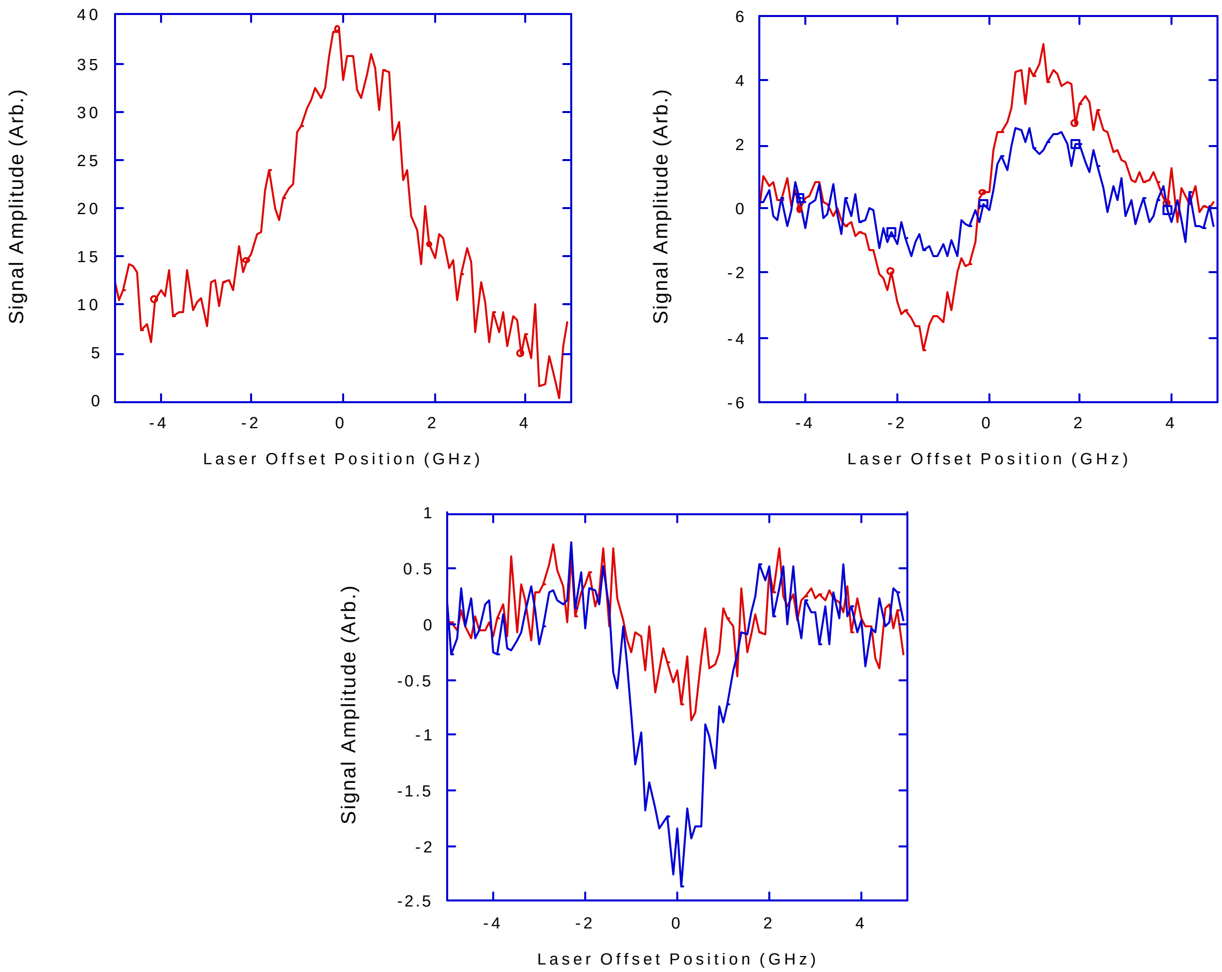

Figure 10: FFT spectrum information obtained from the averaging method a) $f_0(v)$ the dc offset, b) real (red) and imaginary (blue) $f_1(\mathbf{r},v,t)$ at 35.5 kHz, and c) real (red) and imaginary (blue) $f_2(\mathbf{r},v,t)$ at 70 kHz.

The red band in Figure 9 corresponds to the dc component of the FFT and is shown in Figure 10a. This is the zeroth order velocity distribution function for the ions. Figure 10b shows the real and imaginary parts of the $f_1(\mathbf{r},v_y,t)$ at the driving frequency. Figure 10c also shows a phase coherent signal at the first harmonic of the driving frequency. To understand exactly what this perturbed velocity distribution is, extra theory is needed. This may be a second order perturbation to the velocity distribution due to wave-wave coupling or it may just be $f_1(\mathbf{r},v_y,t)$ of a wave excited at the harmonic frequency. So, the theory for a perturbed velocity distribution due to wave-wave coupling is needed to compare it to the $f_1(\mathbf{r},v,t)$theory for the perturbed velocity distribution at the harmonic frequencies to determine which type of perturbation they are. The combination of these three graphs shows the power of this method for measuring perturbed velocity distributions. With a single measurement, all of the information needed to analyze the waves in the plasma is obtained; the velocity distribution function, the first order perturbed velocity distribution function, and any other phase coherent perturbations to the velocity distribution function.

4.3 Possibilities of Using Crosspower Spectrum

Two methods for measuring perturbed velocity distributions have been discussed: using a lock-in and using digitized, averaged, time series. However, both methods require a single frequency reference signal to identify the perturbed velocity distributions. It may be possible to measure the perturbed velocity distributions without a single frequency reference signal by measuring the crosspower spectrum between a reference signal and a LIF PMT signal. Now, as with the averaged time series, the real and imaginary parts of the crosspower spectrums could be added together so that frequency components of the two signals that are more strongly correlated and have larger Fourier amplitudes will become largest for a given number of summed crosspower spectrums. Then, plotting the real and imaginary amplitudes as a function of velocity for a single frequency will give the perturbed velocity distributions. This method will be plagued by the same shortcoming of the averaged time series in the sense that a large number of individual crosspower spectrums will be required to increase the signal-to-noise ratio. However, this method should provide a measurement of any phase coherent information at any frequency, as well as show information about signals that have small fluctuations or finite width in frequency. This method could open up perturbed velocity distribution measurements to a wider range of waves using LIF. Also, if differential energy measurements can be made on electron distributions, then this technique could be applied to perturbed velocity distribution measurements of electrons.

Appendix A

The Matlab™ codes in this appendix are the ones needed to produce $f_1(v_y)$. This code includes a drift along B_o and a temperature anisotropy with respect to B_o. The plasma dispersion function is courtesy of DeSouza-Machado and is only valid for real arguments.

Function for calculating $f_1(v_y)$:

```
*************************************************************************

function [ansf1] = f1vyplus(v,a,kz,W,theta,vo,T)

% Written by John Kline
% Date: April 22, 2001
%
% The code is similar to DeSouza-Machados' f1vy code but adds
% temperature anisotropy and drift along the ambient magnetic field.
% This codes also does the f1 summation in a different order the
% Desouza-Machados.

% *This code requires the zf function which is the plasma dispersion
% function.

% Here the variables are

%  a = k_perp*rho          Dimensionless Kperp
%  kz = Wci/(K,, Vth)      Dimensionless 1/Kparallel
%  W = W/Wci               Dimensionless wave freqency
%  theta                   angle between x and y direction in plane perp
%                          to z or Bo
%  T = T,,/Tperp           Temperature ratio
%  vo = Vdrift/Vthperp     Dimensionless drift velocity


max = 14;               % Number of terms for sumations over n,m, and p.
phi = 1.7*10^-12;       % Electrostatic potential


c=a.*sin(theta);        % Dimensionless Ky variable equal to
Kperp*rho*sin(theta)
d=a.*cos(theta);        % Dimensionless Kx variable equal to
Kperp*rho*sin(theta)


kz = kz/sqrt(2);        % Dimensionless Kz variable equal to
Wci/(K,,*Vth*sqrt(2))

vo = vo/sqrt(2);        % Dimensionless Drift velocity

i = sqrt(-1);           % to use for imaginary numbers
```

```
% Here is the sum over n for all m and p.
totn = 0;              % initialize newn
for n=-max:max,
   totm = 0;
   totp = 0;

   % Start of loop over p.  Here only the terms where n-p/2 are integer
   % are valid.  Thus we only calculate a term if (n-p)/2 has no
   % remainder
   % i.e. modulos of (n-p)/2 equals zero
   for p = -max:max,
      if ( mod(n-p,2) == 0 )
           % These are the modified Bessel function terms in f1(vy)
           newp=BESSELI(p,(a*c)).*BESSELI((n-p)/2,(a^2/4)).*exp(-*p*pi/2);
        else
           newp= 0.0;
       end
      totp=totp+newp;  %sum each of the p terms
   end %end loop over p

   %start of loop over m
   for m=-max:max,

      eta = (W -(n+m))*kz-vo;   %argument for plasma dispersion function
      eta0 = W*kz-vo;           %argument for plasma dispersion function
                                %with n=m=0

      %  This term contains all terms with m's in them along with the
      %  plasma.
      %  dispersion function and temperature anisotropy terms.
      newm = (1+zf(eta)*((eta0*T)+(1-T)*eta)).*BESSELJ(m,(a*v))...
             .*exp(-1.*v.^2./2).*exp(i*d*v)...
             .*exp(-i*m*pi/2)*exp(i*(m+n)*theta);

      totm = newm.*totp + totm; % sum each of the m terms
   end %end of loop over m

   totn = totn+totm; % sum each of the n terms
end

% coefficient to the f1 equation.
coef=((1.6.*10.^-19).*phi)./(0.99.*10.^-31);

%return value:  This has the coefficient term and all terms that
%               are not dependent on the summations in f1(vy).
ansf1 = coef.*totn.*exp(-1*a^2/4-1*c^2/2);
```

Plasma Dispersion Function:

DeSouza-Machado's function for the plasma dispersion function. Be careful there is a singularity at 0.0.

****

```
function ans=zf(z_in)
%This is the plasma dispersion function
%calculated by continued fractions

iImagNeg=(imag(z_in) < 0.0);
z1=z_in.*(~iImagNeg);            %imag(z1) > 0 so we are ok
z2=conj(z_in).*(iImagNeg);       %imag(z2) < 0 so be careful
z=z1+z2;

d(1)=-1.7724538509*i;
d(2)=-.6440746838*i;
d(3)=-.5122017546*i;
d(4)=-.4350588634*i;
d(5)=-.3478086516*i;
%
rn(1)=-d(1);
rn(2)=-d(2);
rn(3)=.2198388745*i;
rn(4)=.1724283339*i;
rn(5)=.1458055407*i;
%

on=ones(size(z));
ans=zeros(size(z));
for m=1:5
  j=6-m;
  ans=(rn(j)*z)./(on+ans+z*d(j));
  end


  ans=ans./z;


%this is the old code
%ans=real(ans)+rn(1)*exp(-z.^2);

%this is the more general case: force the imag part to be positive
realans=real(ans);
imagans=abs(imag(ans));
ans1=realans+sqrt(-1)*imagans;   %imag(z) forced positive
ans=ans1;

ans1=ans1.*(~iImagNeg);

sqrexponent=-z_in.*z_in;
largesqrexponent=-1;
if ((real(sqrexponent)) > 600.0)
  largesqrexponent=1;
  end
```

```
if (largesqrexponent < 0)
  zsqrexp=exp(-z_in.*z_in);
  %now depending on the sign of imag(z), do some adjustments
  ans2=(conj(ans)+2.0*sqrt(-1)*sqrt(pi)*zsqrexp).*iImagNeg;
  ans=ans1+ans2;
else
  ans2=(conj(ans)+2.0*sqrt(-1)*sqrt(pi)*1.0e100).*iImagNeg;
  ans=ans1+ans2;
  end
```

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